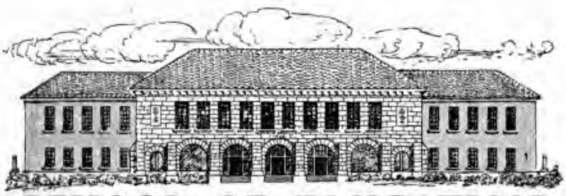
ANALYTIC GEOMETRY.

H. T. EDDY.

PLANE



H. E. Pease
193

A

TREATISE

ON THE

PRINCIPLES AND APPLICATIONS

OF

ANALYTIC GEOMETRY.

BY

HENRY T. EDDY, C.E., PH.D.,
PROFESSOR OF MATHEMATICS AND ASTRONOMY IN THE UNIVERSITY
OF CINCINNATI.

PHILADELPHIA:
COWPERTHWAIT & COMPANY.
1874.

623261

WESTCOTT & THOMSON,
Stereotypers and Electrotypers, Phila.

EDMUND DEACON,
Printer, Phila.

PREFACE.

The following treatise, designed as a text-book upon Analytic Geometry, has been written with the most practical ends in view, and is intended to meet the wants of classes in Scientific and Technological Schools, Colleges and Universities. While the needs of the student of Mechanics, Astronomy and Civil Engineering have never been forgotten, it has been found possible to so select the material and to put it in such shape as to adapt the work to the student who pursues the subject merely as a part of a liberal education.

The prime difficulty the ordinary student meets in the study of analytic geometry is in the use of *variables,* since with these he has had no previous acquaintance.

No pains has been spared to make the introduction to their use clear and free from all other complexities. To this end a thorough knowledge of co-ordinates has been first secured by the study of the relations of points, the transformation of co-ordinates, etc.

Again, the entire subject of the general relation of constant and variable quantities is postponed to Chapter V, at which point the student will have attained a sufficient acquaintance with the processes and notation peculiar to analytic geometry to grasp the ideas advanced and use them in after work.

To secure an accurate knowledge of the meaning of the general equations, it is essential that the student should solve numerous numerical examples. They should be illustrations, and of such simple character as to be readily solved by any one who understands the preceding text.

Such are the examples interspersed through the work, which should in no case be omitted. Indeed, if the class is numerous, the teacher is advised to largely increase the number of examples as class-room work by substituting other numbers than those used, and giving each example to a sufficient number of different computers to ensure correct results.

The *Exercises* are much more difficult than the examples, and have two objects in view: first, as original work for the more ambitious students; and secondly, as results to be referred to in the students' subsequent studies. They may be omitted by the ordinary student.

The great difficulty which the teacher experiences is not usually that the

student cannot be made to apprehend the true import of the demonstrations, but it is this—that he afterward fails to recall the necessary equations and their significance.

To assist the teacher in this vital point, the statement of each theorem is in a form to be memorized, and contains some important equation and its signification. The importance of acquiring a perfect familiarity with these statements in algebraic language instead of ordinary language cannot be too strongly emphasized. It has been found by the best teachers that ten or fifteen minutes during every recitation hour should be spent in reciting from memory the statements of all theorems previously studied.

The form of notation adopted is thoroughly systematized, and prepares the student to read with ease the great modern writers upon analytic geometry. The marked value of the angular notation used is a sufficient recommendation for its adoption. For it I am happy to acknowledge my indebtedness to Prof. J. M. Peirce, of Harvard University, from whose works it is borrowed.

The one great defect of text-books upon analytic geometry is the omission of general principles. It appears to be assumed that an acquaintance with its ordinary processes gives the student a knowledge of its principles. This is far from being a correct assumption, as an examination of the general principles stated and demonstrated in Chapter V will abundantly show.

The general discussion of curves and their singularities by means of their approximate equations—a method due to the genius of Newton—is here for the first time rendered accessible to the ordinary student, and it is thought that it will serve a most useful purpose by putting into his hands an instrument of research of practical value whose power compares favorably with the resources of the differential calculus.

No attempt has been made in the last chapters to follow the beaten track of previous text-books, but rather to select matter respecting spirals, etc., of the greatest use to the student.

The book will be found to be suited to the wants of classes taking either a longer or shorter course by the various omissions indicated in the course of it.

I take this opportunity to express my thanks to Prof. James Edward Oliver, of Cornell University, for many happy suggestions.

I am especially indebted to Prof. E. W. Hyde, formerly of Cornell University, who has with signal ability and fidelity assisted me in preparing the book for the press. In particular, the *Examples* were, almost without exception, made by him.

Part Second, upon Solid Geometry, is in preparation, and will be issued at as early a date as circumstances may permit.

HENRY T. EDDY.

PRINCETON, NEW JERSEY,
August 15, 1874.

CONTENTS.

CHAPTER VI.

The Parabola.

CHAPTER VII.

The Hyperbola and Ellipse.

CHAPTER VIII.

General Equation of the Second Degree.

CHAPTER IX.

Curves of the Third and Fourth Degrees.

CHAPTER X.

Higher Algebraic Curves.

CHAPTER XI.

Transcendental Curves.

CHAPTER XII.

Spirals and Polar Curves.

Abbreviations and Signs.

E. G. is used to introduce an illustrative example.

N. B. is used to introduce some useful notation or convention adopted.

A *period* may signify *multiplied by*, and a *colon* may signify *divided by*.

$\pi = 3.14159$ is the semi-circumference of the circle whose radius is unity.

$\because$ signifies *therefore*, and —*i. e.* signifies *that is*.

OPQ may signify *the angle OPQ*, etc.

The dagger (†) signifies that the proposition to which it is affixed may be omitted if desirable.

$\phi(x, y)$ signifies some unknown function of x and y, and is read *phi function* of x and y.

Read combinations of subscripts, primes and powers as follows: P_1, *pe one;* P_2, *pe two;* P', *pe prime;* P'', *pe second;* P_1'', *pe one second;* P'_2, *pe prime two;* x'^2, *ex prime square;* x_2^2, *ex two square*, etc.

$\overset{y}{x}$ signifies *the angle between x and y* (Art. 12).

Greek Alphabet.

α alpha.	ι iota.	ρ rho.
β beta.	κ kappa.	σ ς sigma.
γ gamma.	λ lambda.	τ tau.
δ delta.	μ mu.	υ upsilon.
ϵ epsilon.	ν nu.	φ ϕ phi.
ζ zeta.	ξ xi.	χ chi.
η eta.	o omicron.	ψ psi.
ϑ θ theta.	π pi.	ω omega.

INTRODUCTION.

Analytio Geometry is distinguished from other geometry* in this one particular: *its investigations are conducted by means of the forms and symbols of algebra.*

Co-ordinate Geometry is that branch of Analytic Geometry in which investigations are conducted by means of *co-ordinates.*

Co-ordinates are quantities which determine *position.* Position can be determined only by reference to some assumed position.

Latitude and longitude are co-ordinates. They determine the position of a point on the earth by reference to the equator, and to an assumed meridian.

Time by the clock is also a co-ordinate. It determines a point of time by reference to mean noon.

---

* In this treatise the elementary propositions of geometry and trigonometry are assumed to be already proven.

ANALYTIC GEOMETRY.

PART FIRST.

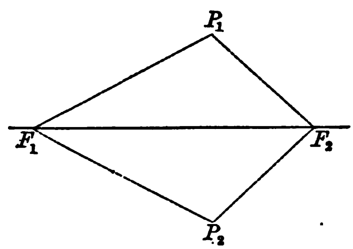
PLANE GEOMETRY.

CHAPTER I.

CO-ORDINATES.

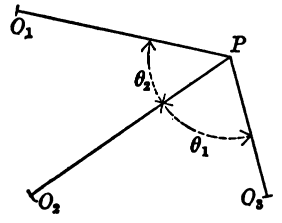
1. The position of a point in a plane may be determined by co-ordinates of many different kinds. A few of these systems are as follows.

2. Focal co-ordinates.—From the given or assumed points F_1 and F_2, let the point P_1 be at the distances r_1 and r_2 respectively. Then r_1 and r_2 are the *focal co-ordinates* of P_1. It is to be noticed that r_1 and r_2 are also the co-ordinates of a *second point* P_2. The two triangles $P_1F_1F_2$ and $P_2F_1F_2$ are evidently equal, since their corresponding sides are equal, and hence the points

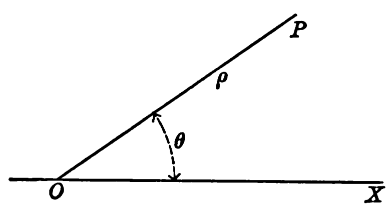


P_1 and P_2 are *symmetrically situated* with reference to the right line F_1F_2, joining the points F_1F_2. The points F_1 and F_2 are called *foci*, and the right line joining them is called an *axis*. This system of co-ordinates is at present of limited application.

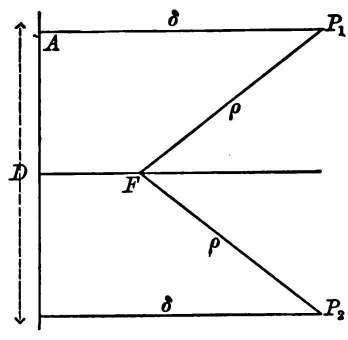
3. Angular co-ordinates.—From the given or assumed points $O_1 O_2 O_3$ let the right lines to P contain the angles θ_1 and θ_2; then θ_1 and θ_2 are the *angular co-ordinates* of P. This system is used in the topographic work of the United States Coast Survey.



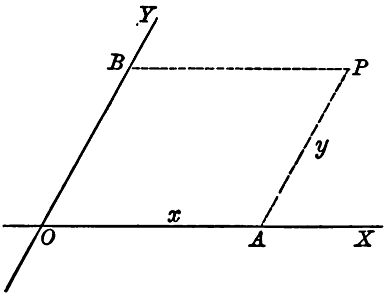
4. Polar co-ordinates.—From the given or assumed point O upon the given or assumed line OX, let OP have a length ρ, and make an angle θ with OX; then ρ and θ are the *polar co-ordinates* of P. This system of co-ordinates is much used in ordinary analytic work.



5. Linear co-ordinates.—If from the point P_1 we let fall the perpendicular AP_1 upon the given or assumed line D, and let it have the length δ, and also draw from the given or assumed point F the right line FP_1 with the length ρ; then ρ and δ are the *linear co-ordinates* of P_1. The line D is called a *directrix*, and the point F a *focus*. If a line be drawn through F *perpendicular* to D, it will be an *axis*. It will be observed that ρ and δ are also the co-ordinates of a *second point* P_2, and that P_1 and P_2 are *symmetrically situated* with reference to this axis. This system of co-ordinates is employed to some extent in this work, in the treatment of the conic.

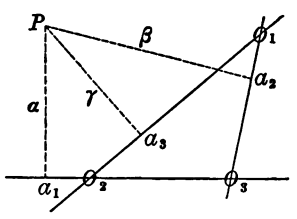


6. Bilinear or Cartesian co-ordinates.—From P draw two lines PB and PA respectively parallel to two given or assumed right lines OX and OY; then $OA=x$ and $OB=y$ are the *Cartesian co-ordinates* of P.



When $XOY=90°$, the system is called rectangular, and is the system principally employed in this treatise.

7. Trilinear co-ordinates.—From P let fall three perpendiculars $\alpha\beta\gamma$ upon the sides of a given or assumed triangle $O_1O_2O_3$; then $\alpha\beta\gamma$ are the *trilinear co-ordinates* of P. This system, together with its reciprocal system, *tangential co-ordinates*, is used in modern analytic investigations.



POSITIVE AND NEGATIVE.

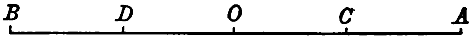
8. Motion is of two kinds, that of *translation*, and that of *rotation*.

The **distance** between two points is the *amount which a point must move* in passing from one of the points to the other.

If distance, or motion of translation of a point in *one direction* be assumed to be *positive*, then motion in the *opposite direction* will be *negative*.

Distance or length is measured in feet, inches, metres, or some other arbitrary unit.

9. If the distance from O to A is OA, and that from A to O is AO; then $OA = -AO$.



If $BO = OA$,

then $OA = -OB$, and $AB = -BA$.

Also $OA + AC = OA - CA = OC$,

or $BO + OD = BO - DO = BD$,

and $OD + DC + CA = -DO + DO + OC + CA = OA$.

$\therefore$ $OA = OV + VV_1 + V_1V_2 + \ldots + V_{n-1}V_n + V_nA$,

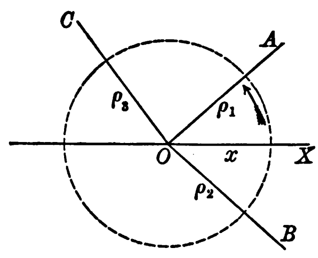
in which V, V_1, V_2, etc., are *any points whatever* in the line AO.

10. The **angle** between two lines is the *amount which a line must turn* in passing from one of the lines to the other.

If the angle or rotation of a line in *one direction* be assumed to be *positive*, then motion in the *opposite direction* will be *negative*.

Angle or rotation is measured in degrees, grades or arbitrary fractions of an entire rotation. The Greek letter π is commonly used to designate the length of the semicircumference of the circle whose radius is unity.

11. If the angle from OX to OA is XOA, and that from OA to OX is AOX, then $XOA = -AOX$. If $BOX = XOA$, then $XOA = -XOB$, and $BOA = -AOB$. The direction indicated by the *arrow* is the direction which will be considered *positive*.



12. *N. B.*—If the direction OX be called x, for convenience, and OA, ρ_1, then the angle XOA may be written $\overset{\rho_1}{x}$,* and will be read, "*the angle between x and* ρ_1."

* This must be carefully distinguished from a fractional expression. The *lower letter* is *written first* and *read first*.

Then AOX will be written $\frac{x}{\rho_1}$.

Since $XOA = -AOX$, we have $\frac{\rho_1}{x} = -\frac{x}{\rho_1}$.

Also $\qquad \frac{\rho_3}{x} + \frac{\rho_1}{\rho_3} = \frac{\rho_3}{x} - \frac{\rho_3}{\rho_1} = \frac{\rho_1}{x};$

or $\qquad \frac{\rho_3}{\rho_2} + \frac{\rho_1}{\rho_3} = \frac{\rho_1}{\rho_2};$

$\therefore \qquad \frac{\rho_n}{x} + \frac{\rho_m}{\rho_n} + \frac{\rho_r}{\rho_m} + \frac{\rho_1}{\rho_r} = \frac{\rho_1}{x},$

in which ρ_n, ρ_m and ρ_r have any directions whatever. Therefore the equation $\frac{\alpha}{x} + \frac{\beta}{\alpha} + \frac{\gamma}{\beta} + \frac{\delta}{\gamma} + \frac{\rho_1}{\delta} = \frac{\rho_1}{x}$ expresses the final angle between x and a line turning from x successively to $\alpha\ \beta\ \gamma\ \delta$ and ρ_1.

It is to be noticed that all parallels have the same direction, so that any line has the same direction as a parallel to it through O; from which it follows that the sum of the exterior angles of any convex polygon is equal to 360°, as is also proved in geometry.

It is also to be noticed that any multiple of $2\pi = 360°$ may be neglected, since a complete revolution does not change the position of a line. Also that $\frac{-x}{x} = 180°$.

13. Since $\frac{y}{x} = -\frac{x}{y}$, in which x and y have any directions whatever, we have by trigonometry:

$$\sin\frac{y}{x} = \sin\left(-\frac{x}{y}\right) = -\sin\frac{x}{y}$$

$$\cos\frac{y}{x} = \cos\left(-\frac{x}{y}\right) = +\cos\frac{x}{y}$$

$$\tan\frac{y}{x} = \tan\left(-\frac{x}{y}\right) = -\tan\frac{x}{y}$$

$$\cot\frac{y}{x} = \cot\left(-\frac{x}{y}\right) = -\cot\frac{x}{y}$$

$$\sec\frac{y}{x} = \sec\left(-\frac{x}{y}\right) = +\sec\frac{x}{y}$$

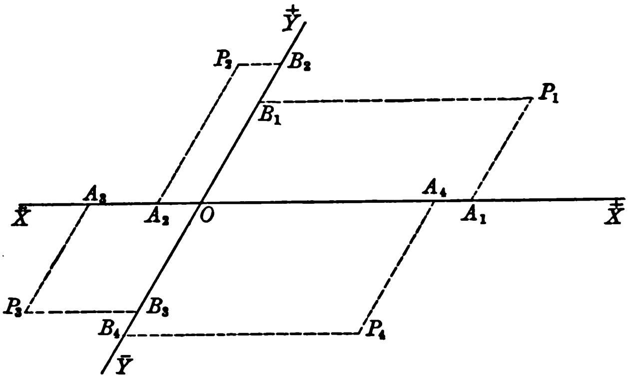
$$\operatorname{cosec}\frac{y}{x} = \operatorname{cosec}\left(-\frac{x}{y}\right) = -\operatorname{cosec}\frac{x}{y}$$

CHAPTER II.

THE POINT.

CARTESIAN CO-ORDINATES.

14. Abscissa, or x. The distance $OA_1 = B_1P_1$ is called the *abscissa* of P_1, or the x *co-ordinate* of P_1, or simply the x of P_1.



When this co-ordinate is measured to the *right* of O, it is usually reckoned *positive*, and when measured to the *left*, *negative.*

15. Ordinate, or y. The distance $OB_1 = A_1P_1$ is called the *ordinate* of P_1, or the y *co-ordinate,* or simply the y of P_1.

It is usual to consider y *positive above,* and *negative below* O; *e.g.*, for the point P_3, x and y are both negative.

16. Axes.—The lines OX and OY are called the *axes of reference,* or the *co-ordinate axes,* or simply the *axes* of x and y.

The axis of x is usually taken horizontal.

17. Origin.—The point O, in which the axes of x and y intersect, is called the *origin* of *co-ordinates*, or simply the *origin*.

The angle $\overset{+}{X} O \overset{+}{Y}$ is called the *first* angle.

" " $\overset{-}{X} O \overset{+}{Y}$ " " *second* "

" " $\overset{-}{X} O \overset{-}{Y}$ " " *third* "

" " $\overset{+}{X} O \overset{-}{Y}$ " " *fourth* "

For a point situated

in the *first* angle, x is $+$ and y is $+$.

in the *second* angle, x is $-$ and y is $+$.

in the *third* angle, x is $-$ and y is $-$.

in the *fourth* angle, x is $+$ and y is $-$.

18. Oblique Axes.—When the angle between the axes of x and y is *not* $90°$, the system is called *oblique*.

It will be convenient to use $\omega = \overset{+}{X} O \overset{+}{Y}$ to denote this angle.

Rectangular Axes.—When $\omega = 90°$, the system is called *rectangular*.

19. ***N. B.***—We shall use x and y to denote the co-ordinates of *any point* P with reference to the axes OX and OY.

We shall use x' and y' to denote the co-ordinates of any point P with reference to other axes, $O'X'$ and $O'Y'$, and x'' and y'' with reference to $O''X''$ and $O''Y''$, etc., etc.

We shall also for convenience use x_1 and y_1 to denote co-ordinates of some particular point P_1, *i. e.*, x_1 and y_1 are particular values of x and y. Also x_2 and y_2, for the point P_2, are other particular values of x and y, etc., etc.

Similarly, x_1' and y_1', or x_2' and y_2' are particular values of x' and y'.

Proposition 1.

20. *Theorem.*—*The equations*

$$x = x_1 \quad \text{and } y = y_1$$

represent a point; in which x and y may be the bilinear co-ordinates of any point, and x_1 and y_1 are their values for this particular point.

For a single point has position only, and its position is completely determined by these equations.

N. B.—We shall frequently speak of the point (x_1, y_1), or $(-2, 3)$, etc., meaning the point whose co-ordinates are

$$x = x_1, \quad y = y_1, \quad \text{or } x = -2, \quad y = 3, \text{ etc.}$$

21. *Corollary.*—*The equations* $x = 0$ and $y = 0$ *represent the origin.*

22. *Examples.*—Locate the points represented by the following equations, both in *rectangular* and *oblique* co-ordinates, using first ½ in., then ⅓ in., as the unit of measure.

(1.) $x = 2,\qquad y = 5.$

(2.) $x = -1,\qquad y = 3.5.$

(3.) $x = -2,\qquad y = -1.$

(4.) $x = \frac{5}{2},\qquad y = -\frac{2}{3}.$

(5.) $x + y = 1,\qquad x - y = 2.$

(6.) $x + 3y - 4 = 0,\qquad 2x - 7y + 1 = 0.$

(7.) $x + y = 1,\qquad x^2 - y^2 = 0.$

(8.) $x = 1,\qquad x^2 + xy + y^2 = 1.$

Proposition 2.

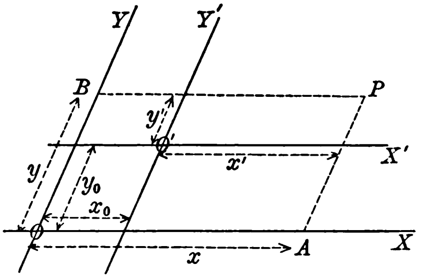
23. Theorem.—*The equations*

$$x = x_0 + x', \text{ and } y = y_0 + y',$$

are the equations by which any point is referred to a new system of axes parallel to the primitive system; in which x and y are the primitive co-ordinates of any point, x' and y' the new co-ordinates of the same point, and x_0 and y_0 are the co-ordinates of the new origin.

The equations are evidently true from inspection of the figure.

For, $OA = x = x_0 + x'$, and $OB = y = y_0 + y'$.



24. Schol.—By Art. 23,

$$x' = x - x_0,$$

and $y' = y - y_0$.

It is evident that x_0 and y_0 are positive when measured from the origin O, but negative when measured from O',

$$\therefore \quad x_0 = -x_0', \text{ and } y_0 = -y_0'.$$

Substituting in the previous equations, we have

$$x' = x_0' + x, \text{ and } y' = y'_0 + y,$$

$$\therefore \quad x = x' - x_0', \text{ and } y = y' - y_0'.$$

These axes may be either *oblique* or *rectangular.*

The above change of axes of reference is a case of *transformation of co-ordinates.* This change will evidently not affect the relations of the point to any other points or lines than the origin and axes.

25. *Examples.*—Construct the axes and points in the following examples (see Art. 18):

(1.)	(2.)	(3.)	(4.)
$x_0 = 2,$	$x_0 = -3,$	$x = 5,$	$x = -7,$
$y_0 = 5,$	$y_0 = 1,$	$x' = 3,$	$y = -6,$
$x' = -4,$	$x' = 3,$	$y' = 2,$	$x_0 = 3,$
$y' = 3,$	$y' = -5,$	$y_0 = -4,$	$y_0 = -2,$
$\omega = 45°.$	$\omega = 90°.$	$\omega = 120°.$	$\omega = 60°.$

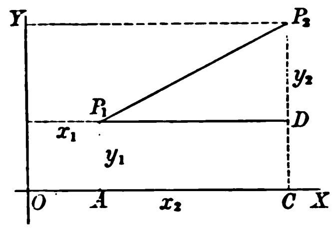
(5.) The three vertices of a triangle are the points, $(2, 1)$, $(3, -4)$, $(-3, -2)$, and the co-ordinates of the new origin are $x_0 = 5$, $y_0 = -4$; construct the triangle, and find the new co-ordinates of the vertices, when $\omega = 90°$: also when $\omega = 120°$.

Proposition 3.

26. *Theorem.*—*The equation*

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

expresses the distance between two points; in which r is the distance, and (x_1, y_1) and (x_2, y_2) are the rectangular co-ordinates of the points.



Since the square on the hypotenuse is equal to the sum of the squares on the two sides of a right-angled triangle, the equation is evidently true from inspection of the figure. For

$$\overline{P_1P_2}^2 = \overline{P_1D}^2 + \overline{P_2D}^2 = (OC - OA)^2 + (CP_2 - CD)^2;$$

$$\therefore \quad r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

27. *Cor.*—If $x_1 = 0$, and $y_1 = 0$ (Art. 21), then $r = \sqrt{x_2^2 + y_2^2}$.

It is to be noticed that this proposition is *general*, though proved only in the first angle.

28. ***Schol.***—The expression

$$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2},$$

when transformed to new parallel axes by the equations (Art. 23)

$$x=x_0+x', \quad y_0=y+y',$$

is $$r=\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}.$$

For $$x_2-x_1=[x_0+x_2'-(x_0+x_1')]=x_2'-x_1',$$

and similarly $$y_2-y_1=y_2'-y_1'.$$

It is to be noticed this transformation does not affect the relation of the points to each other.

29. ***Examples.***—Find the distances between the following points:

(1.) $(3, -1)$, and $(8, 5)$. *Ans.* 7.81.

(2.) $(-2, -4)$, and $(1, -2)$.

(3.) $(3, 7)$, and $(-3, 5)$.

(4.) Refer the points in Ex. 1 to new parallel axes, the co-ordinates of the new origin being $x_0=6$, $y_0=2$.

(5). In Ex. 3 refer to new parallel axes, making $x_0=3$ and $y_0=7$.

30. ***Exercise.***—Prove that when the axes are oblique,

$$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+2\,(x_2-x_1)\,(y_2-y_1)\cos\omega.}$$

Proposition 4.†

31. ***Theorem.***—*The equation*

$$y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)=\pm 2t$$

expresses twice the area of the triangle whose vertices are the three points, of which the rectangular co-ordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3), ***and of which the area is*** t.

† All propositions or other divisions of the book marked (†) may be omitted on first reading.

For, $t = \frac{1}{2}\, a_1 a_2\, (a_1 b_1 + a_2 b_2)$

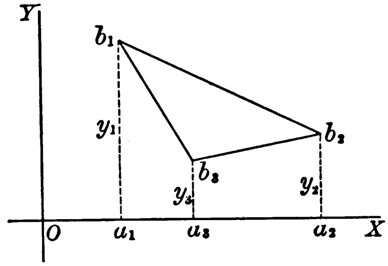
$$- \tfrac{1}{2}\, a_1 a_3\, (a_1 b_1 + a_3 b_3)$$

$$- \tfrac{1}{2} a_3 a_2\, (a_3 b_3 + a_2 b_2).$$

$$\therefore\ 2t = (x_2 - x_1)\,(y_2 + y_1)$$

$$- (x_3 - x_1)\,(y_3 + y_1)$$

$$- (x_2 - x_3)\,(y_2 + y_3).$$



And by multiplying out and canceling we obtain,

$$y_1\,(x_2 - x_3) + y_2\,(x_3 - x_1) + y_3\,(x_1 - x_2) = \pm\, 2t.$$

32. *Cor.*—In the same manner we can obtain the area of *any* polygon. The area of a quadrilateral $= q$ is given by the equation,

$$y_1\,(x_2 - x_4) + y_2\,(x_3 - x_1) + y_3\,(x_4 - x_2) + y_4\,(x_1 - x_3) = \pm\, 2q.$$

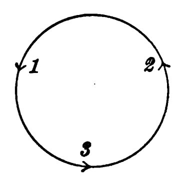
33. *Schol. 1.*—If the preceding points be referred to new axes parallel to the primitive, we shall have (Art. 23),

$$y_1'\,(x_2' - x_3') + y_2'\,(x_3' - x_1') + y_3'\,(x_1' - x_2') = \pm\, 2t,$$

since all the terms that contain x_0 and y_0 will cancel.

The relation between the points is unchanged by the transformation.

34. *Schol. 2.*—It is useful to notice the *cyclic symmetry* of the above expression—that is, that the subscripts follow around in the same cyclic order, viz., 123, 231, 312.



35. *Examples.*—Find the area of the surfaces inclosed by right lines joining the points whose co-ordinates are given in the following examples:

(1.) $(2, 3)$, $(4, 1)$ and $(5, 6)$. *Ans.* 6.

(2.) $(-5, 1)$, $(-2, -3)$, $(4, 6)$. *Ans.* $25\frac{1}{2}$.

(3.) $(-4, 7)$, $(6, 3)$, $(2, -1)$, $(-3, -2)$. *Ans.* 51.

36. *Exercise.*—Prove that when the axes are oblique,

$$[y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)] \sin \omega = \pm 2t.$$

Proposition 5.

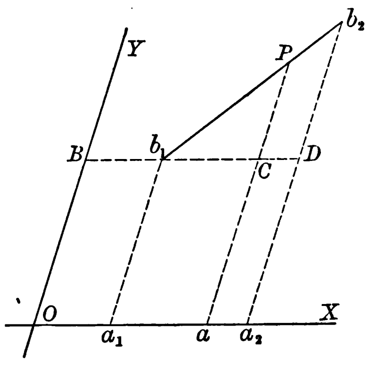
37. *Theorem.*—*The equations*

$$x = \frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}, \text{ and } y = \frac{n_1 y_2 + n_2 y_1}{n_1 + n_2}$$

express the position of a point dividing the distance between two points in a given ratio; in which $n_1 : n_2$ ***is the ratio, and*** (x, y) ***are the co-ordinates of the point dividing the distance between*** (x_1, y_1) ***and*** (x_2, y_2) ***in the given ratio.***

For, let $Oa = x$, $Oa_1 = x_1$ and $Oa_2 = x_2$.

By similar triangles we have



$$b_1C : CD :: b_1P : Pb_2,$$

or $\quad x - x_1 : x_2 - x :: n_1 : n_2.$

$$\therefore \quad x = \frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}.$$

Similarly $y = \dfrac{n_1 y_2 + n_2 y_1}{n_1 + n_2}.$

38. *Schol. 1.*—If the preceding points be referred to new axes parallel to the primitive, since (Art. 23),

$$x - x_1 = x_0 + x' - (x_0 + x_1') = x' - x_1', \text{ etc.};$$

$$\therefore \quad x' - x_1' : x_2' - x' :: n_1 : n_2;$$

$$\therefore \quad x' = \frac{n_1 x_2' + n_2 x_1'}{n_1 + n_2}, \text{ and } y = \frac{n_1 y_2' + n_2 y_1'}{n_1 + n_2}.$$

This transformation does not change the mutual relation of the three points.

39. ***Schol. 2.***—Were the point not situated between (x_1, y_1) and (x_2, y_2), but on either side of these points, n_2 would be negative, and the equations would become

$$x = \frac{n_1 x_2 - n_2 x_1}{n_1 - n_2}, \text{ and } y = \frac{n_1 y_2 - n_2 y_1}{n_1 - n_2}.$$

40. ***Examples.***—(1.) Find the co-ordinates of the point which divides in the proportion of 3 to 5 the line joining the points

$$\left\{\begin{matrix} x+2y=0 \\ \frac{x}{2}+\frac{y}{3}=1 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} x=y-1 \\ y=2x-1 \end{matrix}\right\} \quad Ans. \left\{\begin{matrix} x=2\frac{5}{8} \\ y=\frac{3}{16}. \end{matrix}\right.$$

(2.) Given a line joining two points $(5, -2)$ and $(-3, 7)$, to find the co-ordinates of a point on the line *not between* the given points, which shall divide the line in the proportion of *1* to *7*.

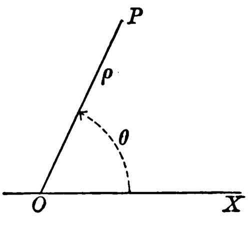
Ans. $x = 6\frac{1}{3}, y = -3\frac{1}{2}$.

41. ***Exercise.***—Prove this proposition also as a case of Prop. 4, when $t = 0$.

POLAR CO-ORDINATES.

42. Radius Vector.—The distance OP is called the *radius vector, or ρ co-ordinate,* or simply the ρ of P.

It is usual to consider ρ as positive when there is no reason for taking it negative.



43. Variable Angle.—The angle XOP is called the *variable angle, or θ co-ordinate,* or simply the θ of P.

It is usual to measure this angle from XO around in a direction *opposite* to the motion of the hands of a watch for *positive* rotation, and *with* the hands for *negative* rotation.

It is often convenient to denote the angle between x and ρ, as $\frac{\rho}{x} = \theta$, which must be carefully distinguished from a *ratio* or *fraction.* (Art. 12.)

44. Pole.—The point O in which all the *radii vectores* intersect is called the *pole.* The pole is the *origin of distance.*

Initial Line.—The line OX from which the variable angle is measured is called the *initial line.* The initial line is the *origin of direction.*

45. ***N. B.***—We shall use ρ and θ to denote general values of the polar co-ordinates referring to a primitive pole and initial line, and ρ' and θ' referring to a new pole and initial line. We shall also use ρ_1 and θ_1, ρ_2 and θ_2, etc., to denote restricted values. (Compare Art. 19.)

Proposition 6.

46. ***Theorem.***—*The equations*

$$\rho = \rho_1 \text{ and } \theta = \theta_1$$

represent a point; in which ρ and θ may be the polar co-ordinates of any point, and ρ_1, θ^1, are their values for this particular point.

For, a single point has position only, and its position is completely determined by these equations.

N. B.—We shall use (ρ_1, θ_1), or $(5, \frac{\pi}{4})$, etc., to indicate the point whose co-ordinates are $\rho = \rho_1$, $\theta = \theta_1$, or $\rho = 5$, $\theta = \frac{\pi}{4}$, etc.

47. ***Cor.***—The equation $\rho = 0$ represents the pole, and the equation $\theta = 0$, or $\theta = \pi$, or $\theta = 2\pi$, etc., represents the initial line.

48. ***Examples.***—Locate the points whose co-ordinates are given below.

(1.) $\rho = 2, \quad \theta = \frac{\pi}{2}.$

(2.) $\rho = -1, \quad \theta = 45°.$

(3.) $\rho = \frac{\theta}{2}, \quad \theta = 2\pi.$

Proposition 7.

49. *Theorem.*—*The equation*

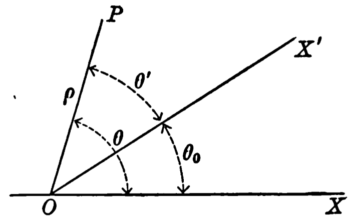
$$\theta = \theta_0 + \theta'$$

is that by which any point is referred to a new initial line through the pole; in which θ_0 is the angle between the new and the primitive initial line.

From inspection of the figure,

$$\frac{\rho}{x} = \frac{x'}{x} + \frac{\rho}{x'}, \text{ or } \theta = \theta_0 + \theta'.$$

Observe that the transformation of this one of the polar co-ordinates of any point, will not change its relations to anything except the initial line itself.



Proposition 8.

50. *Theorem.*—*The equation*

$$r = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}$$

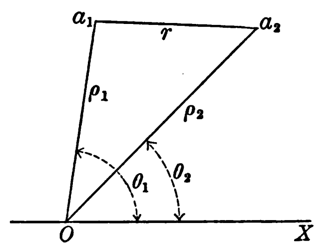
expresses the distance between two points; in which r is the distance between the two points whose polar co-ordinates are (ρ_1, θ_1) and (ρ_2, θ_2).

If $a_1 a_2 = r$, $Oa_1 = \rho_1$, $Oa_2 = \rho_2$,

and $\frac{\rho_1}{\rho_2} = \theta_1 - \theta_2$, we have, by trig.,

$$r^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2).$$

(Compare Art. 30.)



51. *Cor.*—If one of the points, say a_2, be at the origin, $r = \rho$.

52. *Schol.*—If the initial line be changed to coincide with ρ_2,

$$i.\,e., \quad \theta_0 = \theta_2, \quad \therefore \quad \theta_1 - \theta_2 = (\theta_0 + \theta_1') - \theta_0,$$

$$\therefore \quad r = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\theta_1'}.$$

53. ***Examples.***—Find the distances between the points whose co-ordinates are as follows:

(1.) $\rho = 4,\ \theta = \frac{\pi}{2}$, and $\rho = -3,\ \theta = \frac{\pi}{6}$.

Ans. $r = 6.08$

(2.) $\rho = -2,\ \theta = \frac{5\pi}{4}$, and $\rho = 7,\ \theta = \frac{\pi}{4}$.

Ans. $r = 5$.

(3.) $\rho = 8,\ \theta = 350°$, and $\rho = 6,\ \theta = 80°$.

Ans. $r = 10$.

(4.) $\rho = 4,\ \theta = \frac{2\pi}{9}$, and $\rho = 10,\ \theta = -\frac{\pi}{9}$.

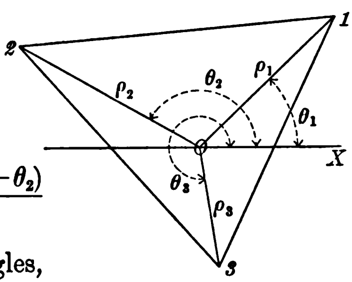
Ans. $r = 8.72$

Proposition 9.†

54. ***Theorem.—The equation***

$$\rho_1 \rho_2 \sin(\theta_1 - \theta_2) + \rho_2 \rho_3 \sin(\theta_2 - \theta_3) + \rho_3 \rho_1 \sin(\theta_3 - \theta_1) = \pm 2t$$

expresses twice the area of any triangle, when t ***is its area, and the polar co-ordinates of its vertices are*** (ρ_1, θ_1), (ρ_2, θ_2), ***and*** (ρ_3, θ_3).



The triangle *123* is equal to

$$O12 + O23 + O31,$$

and by trig. the area of *O12* is

$$\frac{O1 \times O2 \times \sin 1O2}{2} = \frac{\rho_1 \rho_2 \sin(\theta_1 - \theta_2)}{2}$$

and similarly for the other triangles,

$$\therefore\ \rho_1 \rho_2 \sin(\theta_1 - \theta_2) + \rho_2 \rho_3 \sin(\theta_2 - \theta_3) + \rho_3 \rho_1 \sin(\theta_3 - \theta_1) = \pm 2t.$$

This expression is equally true when the pole is not within the triangle.

Observe the cyclic symmetry in the above expression.

55. ***Cor.***—In the same manner we may derive an expression for the area of *any* polygon.

56. ***Examples.***—Find the areas included within right lines joining the points whose co-ordinates are given below.

(1.) $(5, 10°)$, $(2, 100°)$, $(3, 200°)$. *Ans.* $t = 9.26$

(2.) $\left(4, \frac{\pi}{18}\right)$, $\left(5, \frac{2\pi}{9}\right)$, $\left(1, \frac{7\pi}{18}\right)$. *Ans.* $t = 4.52$

(3.) $\left(3, \frac{\pi}{18}\right)$, $\left(5, \frac{2\pi}{9}\right)$, $\left(6, \frac{\pi}{2}\right)$, $\left(2, \frac{\pi}{3}\right)$. *Ans.* $t = 10.21$

Proposition 10.†

57. ***Theorem.***—*The equation*

$$\begin{aligned} &r_1^2 (\rho_1^2 - \rho_2^2)(\rho_3^2 - \rho_1^2) + r_1^2 \rho_1^2 (r_2^2 + r_3^2 - r_1^2) \\ &+ r_2^2 (\rho_2^2 - \rho_3^2)(\rho_1^2 - \rho_2^2) + r_2^2 \rho_2^2 (r_3^2 + r_1^2 - r_2^2) \\ &- r_3^2 (\rho_3^2 - \rho_1^2)(\rho_2^2 - \rho_3^2) + r_3^2 \rho_3^2 (r_1^2 + r_2^2 - r_3^2) = r_1^2 r_2^2 r_3^2 \end{aligned}$$

expresses the relation between the length of the sides and diagonals in any quadrilateral—i. e., the six distances between any four points.

For identically,

$$(\theta_3 - \theta_1) + (\theta_1 - \theta_2) = \theta_3 - \theta_2, \quad \text{or } \alpha + \beta = \gamma,$$

$$\text{if} \quad \theta_3 - \theta_1 = \alpha, \quad \theta_1 - \theta_2 = \beta \quad \text{and } \theta_3 - \theta_2 = \gamma$$

By trig. $\sin\alpha \cos\beta + \cos\alpha \sin\beta = \sin\gamma$

Squaring,

$$\sin^2\alpha \cos^2\beta + 2 \sin\alpha \sin\beta \cos\alpha \cos\beta + \cos^2\alpha \sin^2\beta = \sin^2\gamma$$

$$\therefore \quad (1 - \cos^2\alpha)\cos^2\beta + 2\sin\alpha \sin\beta \cos\alpha \cos\beta + \cos^2\alpha \sin^2\beta = 1 - \cos^2\gamma,$$

$$\therefore \quad \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \cos^2\beta + \cos^2\gamma - 2\cos\alpha \cos\beta(\cos\alpha \cos\beta - \sin\alpha \sin\beta) = 1,$$

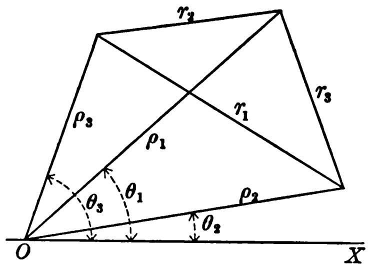
$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha \cos\beta \cos\gamma = 1. \;.\;.\;.\;.\; (a.)$$

By Art. 50,

$$\cos \alpha = \frac{\rho_2^2 + \rho_3^2 - r_1^2}{2\,\rho_2\,\rho_3},$$

$$\cos \beta = \frac{\rho_1^2 + \rho_2^2 - r_3}{2\,\rho_1\,\rho_2},$$

$$\cos \gamma = \frac{\rho_3^2 + \rho_1^2 - r_2^2}{2\,\rho_3\,\rho_1}.$$



Substituting these values in eq. (*a.*), we have the above result.

Proposition 11.†

58. ***Theorem.***—*The equation*

$$\frac{AP\,.\,QB}{AQ\,.\,PB} = \frac{\sin AOP \sin QOB}{\sin AOQ \sin POB} = a \text{ constant}$$

expresses the relation of the four distances between the four points in which any line intersects a pencil of four rays.

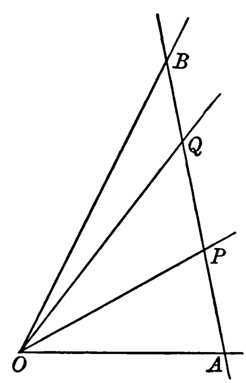
For, if p = length of the perpendicular let fall from O upon AB, we obtain the following expressions for double areas of triangles.

$$p.\,AP = OA\,.\,OP \sin AOP \quad . \; . \; . \quad (a.)$$

$$p.\,QB = OQ\,.\,OB \sin QOB \quad . \; . \; . \quad (b.)$$

$$p.\,AQ = OA\,.\,OQ \sin AOQ \quad . \; . \; . \quad (c.)$$

$$p.\,PB = OP\,.\,OB \sin POB \quad . \; . \; . \quad (d.)$$



The product of equations (*a.*) and (*b.*), divided by the product of equations (*c.*) and (*d.*), gives the above result, which is the same for every line intersecting the pencil.

59. ***Schol.***— $AP \,.\, QB : AQ \,.\, PB$

is called the *anharmonic ratio* of the pencil $O \cdot\cdot APQB$.

The ratios $AP \,.\, QB : AB \,.\, PQ$

and $AB \,.\, QP : AQ \,.\, PB$

are also of constant value, as may be proved in a similar manner.

PROJECTIONS.

60. Projection of a Point.—The foot of a perpendicular let fall from any point upon a line is the orthogonal (*i. e.*, rectangular) *projection of the point upon the line.* Similarly the *oblique* projection of a point may be obtained, but the *orthogonal* projection will always be understood unless otherwise stated.

61. Projection of a Distance.—The distance between the projections of two points is the projection of the distance between the points.

Proposition 12.

62. ***Theorem.***—*The equations*

$$x_2 - x_1 = r \cos {}^{r}_{x}, \quad \text{and } y_2 - y_1 = r \sin {}^{r}_{x}$$

express the projection of the distance between two points upon the rectangular axes of x and y; in which r is the distance between the two points (x_1, y_1) and (x_2, y_2).

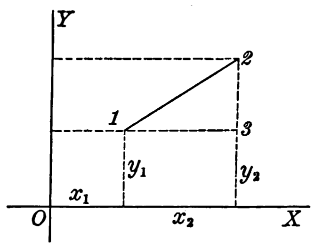
For, if $\overline{1\,3} = x_2 - x_1$, and $\overline{1\,2} = r$,

by trig. $x_2 - x_1 = r \cos 312$;

$\therefore \quad x_2 - x_1 = r \cos {}^{r}_{x}$

Similarly $y_2 - y_1 = r \cos \left(90^\circ - {}^{r}_{x}\right)$

$\therefore$ by trig. $y_2 - y_1 = r \sin {}^{r}_{x}$.



The given distance is the hypotenuse of a right-angled triangle, and its projections on the axes of x and y are respectively parallel and equal to the base and perpendicular.

63. ***Cor.***—If the point *1* coincides with *O*,

$$x_2 = r \cos \frac{r}{x}, \quad \text{and } y_2 = r \sin \frac{r}{x}.$$

64. ***Example.***—Find the projections on the axes of x and y of the distance between two points, when

$$r = 5, \quad \text{and } \frac{r}{x} = 36^\circ\ 52'\ 12''.$$

Ans. $x_2 - x_1 = 4$, and $y_2 - y_1 = 3$.

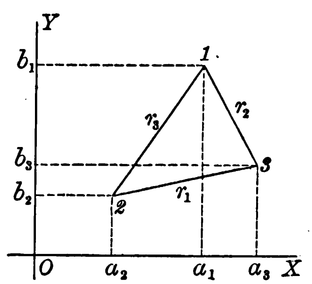
Proposition 13.

65. ***Theorem.—The equations***

$$r_1 \cos \frac{r_1}{x} = r_2 \cos \frac{r_2}{x} + r_3 \cos \frac{r_3}{x}$$

$$r_1 \sin \frac{r_1}{x} = r_2 \sin \frac{r_2}{x} + r_3 \sin \frac{r_3}{x}$$

express the fact that the projection of one side of any triangle upon any lines, as upon the rectangular axes of x and y, is equal to the sum of the projections of the two remaining sides upon the same line; in which r_1, r_2 and r_3 are the three sides of the triangle.



For, $a_2\, a_3 = a_2\, a_1 + a_1\, a_3$

$$\therefore\ r_1 \cos \frac{r_1}{x} = r_2 \cos \frac{r_2}{x} + r_3 \cos \frac{r_3}{x}.$$

Also,

$$b_2\, b_3 = b_2\, b_1 + b_1\, b_3; \quad \text{(Art. 9.)}$$

$$\therefore\ r_1 \sin \frac{r_1}{x} = r_2 \sin \frac{r_2}{x} + r_3 \sin \frac{r_3}{x}.$$

66. ***Cor.***—Since r_2 might be the third side of a new triangle, etc., it follows that the sum of the projections upon the axis of x, of any broken lines leading from *2* to *3*, is equal to the projection of the distance $\overline{2\ 3}$ upon the axis of x. By the word *sum* is to be understood *algebraic sum*, since the projections of any points falling without $a_2\, a_3$, would give us negative distances. (Art. 9.)

67. ***Example.***—Show the truth of the preceding proposition numerically by applying the formula to the following data. Co-ordinates of (1), $x=3$, $y=2$; of (2), $x=6$, $y=1$; of (3), $x=2$, $y=6$;

$$\frac{r_1}{x}=128^\circ\ 39'\ 25'';\quad \frac{r_2}{x}=104^\circ\ 2'\ 10'';\quad \frac{r_3}{x}=161^\circ\ 34'.$$

TRANSFORMATION.

68. The **Transformation** *of the co-ordinates* of any point is the reference of the point to a *new system of co-ordinates.*

N. B. *Review Arts. 12, 13, 18 and 19.*

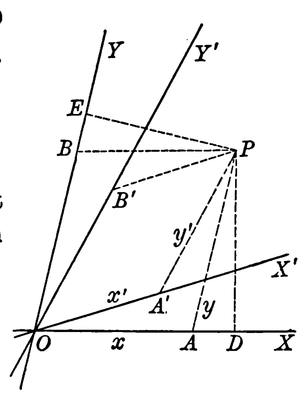
Proposition 14.

69. ***Theorem.***—*The equations*

$$x\sin\frac{x}{y}=x'\sin\frac{x'}{y}+y'\sin\frac{y'}{y}$$

$$y\sin\frac{y}{x}=x'\sin\frac{x'}{x}+y'\sin\frac{y'}{x}$$

are the equations of transformation of any point from a primitive system of oblique bilinear co-ordinates to a new system, also oblique, the origin remaining the same; in which x and y are the primitive co-ordinates of the point, and x' and y' the new co-ordinates of the same point.



Project the figure $OAPA'$ upon PD drawn perpendicular to the axis of x. Then, by Art. 65,

$$y\sin\frac{y}{x}=x'\sin\frac{x'}{x}+y'\sin\frac{y'}{x}.$$

Similarly, if a perpendicular be let fall upon the axis of y, and we can prove that

$$x\sin\frac{x}{y}=x'\sin\frac{x'}{y}+y'\sin\frac{y'}{y}.$$

Notice the symmetry of these formulæ.

70. ***Examples.***—(1.) Refer the point $x = 3$, $y = 5$, when $\frac{y}{x} = 120°$, to new axes in which $\frac{y'}{x'} = 60°$, and $\frac{x'}{x} = 30°$, the origin remaining the same. Ans. $x' = 0.577$, $y' = 4.042$

(2.) Refer the point $x = 2$, $y = -5$, when $\frac{y}{x} = 60°$, to new axes such that $\frac{y'}{x} = 120°$, and $\frac{y'}{x'} = 80°$, the origin remaining the same.
Ans. $x' = -2.64$, $y' = -3.045$

Proposition 15.

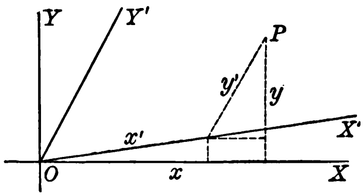
71. ***Theorem.***—*The equations*

$$x = x' \cos \frac{x'}{x} + y' \cos \frac{y'}{x}$$

$$y = y' \sin \frac{y'}{x} + x' \sin \frac{x'}{x}$$

transform from rectangular to oblique axes, the origin remaining the same.

By projections



$$x = x' \cos \frac{x'}{x} + y' \cos \frac{y'}{x}$$

$$y = y' \sin \frac{y'}{x} + x' \sin \frac{x'}{x}.$$

72. ***Cor.***—If the axes of x and x' coincide,

then $$\frac{x'}{x} = 0, \quad \therefore \quad \frac{y'}{x} = \frac{y'}{x'},$$

$$\therefore \quad x = x' + y' \cos \frac{y'}{x'} \left(\text{or } \frac{y'}{x}\right),$$

$$y = y' \sin \frac{y'}{x'} \left(\text{or } \frac{y'}{x}\right).$$

If the axes of y and y' coincide, then

$$\frac{y}{y'} = 0, \quad \text{and } \frac{y'}{x} = 90°;$$

$$\therefore \quad x = x' \cos \frac{x'}{x},$$

$$y = y' + x' \sin \frac{x'}{x}.$$

73. *Examples.*—(1.) Refer the points $(3, 5)$, $(-4, 2)$ when $\genfrac{}{}{0pt}{}{y}{x} = 90°$, to new axes for which $\genfrac{}{}{0pt}{}{y'}{x'} = 30°$, and $\genfrac{}{}{0pt}{}{y'}{x} = 60°$.

Ans. For first point, $x' = 0.196,\ y' = 5.66$
For second point, $x' = -8.93,\ y' = 7.46$

(2.) Refer the point $(3, -4)$ when $\genfrac{}{}{0pt}{}{y}{x} = 90°$ to new axes for which $\genfrac{}{}{0pt}{}{x'}{x} = \frac{\pi}{6}$, and $\genfrac{}{}{0pt}{}{y'}{x'} = \frac{7\pi}{12}$.

Ans. $x' = -0.732,\ y' = -6.175$

74. *Exercise.*—Prove the equations of Art. 71 directly from Art. 69.

Proposition 16.

75. *Theorem.*—*The equations*

$$x \sin \genfrac{}{}{0pt}{}{x}{y} = x' \sin \genfrac{}{}{0pt}{}{x'}{y} + y' \cos \genfrac{}{}{0pt}{}{x'}{y}$$

$$y \sin \genfrac{}{}{0pt}{}{y}{x} = x' \sin \genfrac{}{}{0pt}{}{x'}{x} + y' \cos \genfrac{}{}{0pt}{}{x'}{x}$$

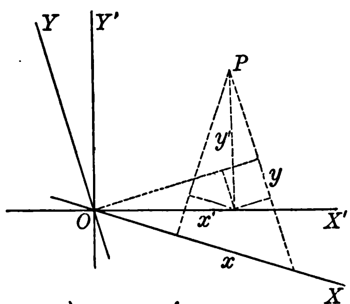
transform from oblique to rectangular axes, the origin remaining the same.

For in the equations of Art. 69,

let $\genfrac{}{}{0pt}{}{y'}{x'} = 90°$, then (Art. 12),

$$\genfrac{}{}{0pt}{}{y'}{y} = \genfrac{}{}{0pt}{}{x'}{y} + \genfrac{}{}{0pt}{}{y'}{x'} = \genfrac{}{}{0pt}{}{x'}{y} + 90°$$

and $\genfrac{}{}{0pt}{}{y'}{x} = \genfrac{}{}{0pt}{}{x'}{x} + \genfrac{}{}{0pt}{}{y'}{x'} = \genfrac{}{}{0pt}{}{x'}{x} + 90°.$



$\therefore$ (by trig.) $\quad \sin \genfrac{}{}{0pt}{}{y'}{y} = \sin \left(\genfrac{}{}{0pt}{}{x'}{y} + 90°\right) = \cos \genfrac{}{}{0pt}{}{x'}{y}$

and $\quad \sin \genfrac{}{}{0pt}{}{y'}{x} = \sin \left(\genfrac{}{}{0pt}{}{x'}{x} + 90°\right) = \cos \genfrac{}{}{0pt}{}{x'}{x}.$

Substitute these values, and we obtain the equations given above.

76. *Cor. 1.*—If the axes of x and x' coincide, then $\frac{x'}{x}=0$, and $\frac{x'}{y}=\frac{x}{y}$.

$$\therefore \quad x \sin\frac{x}{y}=x' \sin\frac{x}{y}+y' \cos\frac{x}{y}$$

$$y \sin\frac{y}{x}=y'.$$

77. *Cor. 2.*—If the axes of y and y' coincide,

$$\text{then} \quad \frac{y'}{y}=0, \quad \text{and} \ \frac{x'}{y}=90^\circ.$$

$$\therefore \quad x \sin\frac{x}{y}=x'$$

$$y \sin\frac{y}{x}=x' \sin\frac{x'}{x}+y' \cos\frac{x'}{x}.$$

78. *Example.*—Refer the point $(-3, 6)$ when $\frac{y}{x}=60^\circ$ to new rectangular axes, such that $\frac{y'}{x}=70^\circ$. *Ans.* $x'=-1.719$, $y'=4.903$

79. *Exercise.*—Prove the equations of Art. 75 from the figure directly.

Proposition 17.

80. *Theorem.*—*The equations*

$$x=x' \cos\frac{x'}{x}-y' \sin\frac{x'}{x}$$

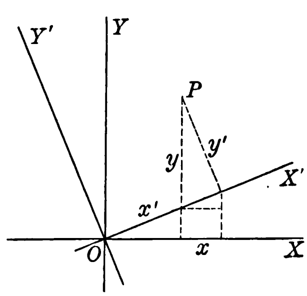
$$y=x' \sin\frac{x'}{x}+y' \cos\frac{x'}{x}$$

transform from rectangular to new rectangular axes, the origin being unchanged.

For, in the equations of Art. 69,

let $$\frac{y}{x}=\frac{y'}{x'}=90^\circ,$$

then $$\frac{y'}{x}=\frac{x'}{x}+\frac{y'}{x'}=\frac{x'}{x}+90^\circ,$$



$$\therefore \quad \sin\frac{y'}{x}=\sin\left(\frac{x'}{x}+90^\circ\right)=\cos\frac{x'}{x}.$$

Also $$\frac{x'}{y}=\frac{x}{y}+\frac{x'}{x}=\frac{x'}{x}-90^\circ,$$

$$\therefore \quad \sin\frac{x'}{y}=\sin\left(\frac{x'}{x}-90^\circ\right)=-\cos\frac{x'}{x}.$$

Moreover, $$\sin\frac{y}{x}=-\sin\frac{x}{y}=1,$$

and $$\sin\frac{x'}{x}=\sin\frac{y'}{y}.$$

Substituting these values in the equations of Art. 69, we have,

$$x = x' \cos {}^{x'}_{x} - y' \sin {}^{x'}_{x}$$

$$y = x' \sin {}^{x'}_{x} + y' \cos {}^{x'}_{x}.$$

This is the most common transformation, and may also be written

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta,$$

θ being the angle through which the rectangular axes are revolved.

81. ***Example.***—Refer the point $(5, -2)$ to new rectangular axes so that we shall have $\theta = 30°$.

Ans. $x' = 3.33,\ y' = 4.23$

82. ***Exercise.***—Prove the equations of Art. 80 directly from the figure.

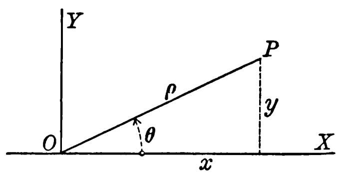
Proposition 18.

83. ***Theorem.***—*The equations*

$$x = \rho \cos {}^{\rho}_{x}, \qquad y = \rho \cos {}^{\rho}_{y} = \rho \sin {}^{\rho}_{x}$$

transform from rectangular to polar co-ordinates, the pole being at the origin, and the axis of x the initial line.

For, ${}^{y}_{x} = 90$, and from Art. 62,



$x = \rho \cos {}^{\rho}_{x}$, and $y = \rho \cos {}^{\rho}_{y}$.

Now $\quad {}^{\rho}_{y} = {}^{x}_{y} + {}^{\rho}_{x} = {}^{\rho}_{x} - 90°.$

$\therefore \quad \cos {}^{\rho}_{y} = \cos \left({}^{\rho}_{x} - 90°\right) = \sin {}^{\rho}_{x}. \quad \therefore \quad y = \rho \sin {}^{\rho}_{x}.$

These equations are often written

$$x = \rho \cos \theta, \text{ and } y = \rho \sin \theta.$$

84. ***Schol.***—*If the initial line is not the axis of x,* we shall have (Art. 49),

$$x = \rho \cos\left(\frac{x'}{x} + \frac{\rho}{x'}\right), \quad \text{and } y = \rho \sin\left(\frac{x'}{x} + \frac{\rho}{x'}\right).$$

The initial line being x'.

85. ***Exercise.***—Prove that the equations

$$x \sin\frac{x}{y} = \rho \sin\frac{\rho}{y'}, \quad \text{and } y \sin\frac{y}{x} = \rho \sin\frac{\rho}{x}$$

transform from oblique to polar co-ordinates.

Proposition 19.

86. ***Theorem.***—*The equations*

$$\rho = \sqrt{x^2 + y^2}, \qquad \sin\frac{\rho}{x} = \frac{y}{\sqrt{x^2 + y^2}}, \qquad \cos\frac{\rho}{x} = \frac{x}{\sqrt{x^2 + y^2}}$$

transform from polar to rectangular co-ordinates.

From Art. 83

$$x^2 = \rho^2 \cos^2\theta, \quad \text{and } y^2 = \rho^2 \sin^2\theta$$

$\therefore$ adding, $\quad x^2 + y^2 = \rho^2(\sin^2\theta + \cos^2\theta)$

$\therefore$ by trig. $\quad x^2 + y^2 = \rho^2, \quad \text{and } \rho = \sqrt{x^2 + y^2}$

also, $\quad \cos\theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \text{and } \sin\theta = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}.$

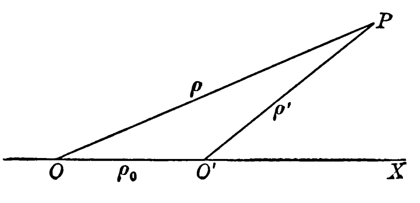
87. ***Example.***—Refer the point whose polar co-ordinates are $\left(5, \frac{\pi}{6}\right)$ to rectangular axes with the origin at the pole, and the axis of x for the initial line. *Ans.* $x = 4.33,\ y = 2.5$

88. ***Exercises.***—Prove from the figure given, that the equations,

$$\rho = \sqrt{\rho'^2 + \rho_0^2 + 2\rho'\rho_0 \cos\frac{\rho'}{x}}$$

$$\rho \sin\frac{\rho}{x} = \rho' \sin\frac{\rho'}{x}$$

$$\rho \cos\frac{\rho}{x} = \rho_0 + \rho' \cos\frac{\rho'}{x}$$



transform from one system of polar co-ordinates to another, in which O is the primitive pole, and O' the new, ρ the old radius vector and ρ' the new, $OO'X$ the initial line, and $OO' = \rho_0$.

The usual method, however, is to transform to rectangular axes, move the origin, and then transform to polar co-ordinates.

Proposition 20.†

89. ***Theorem.***—*The following are the equations of transformation, when the origin is moved to the point* (x_0, y_0), *at the same time that the directions of the axes are changed.*

For by Art. 24 the equations

$$x = x'' - x_0'', \quad \text{and } y = y'' - y_0''$$

will change the axes to a parallel position. On substituting these values of x and y, and then omitting the seconds, as they are not needed longer to distinguish the different systems of axes, we have,

1. The equations

$$(x - x_0) \sin \genfrac{}{}{0pt}{}{x}{y} = x' \sin \genfrac{}{}{0pt}{}{x'}{y} + y' \sin \genfrac{}{}{0pt}{}{y'}{y}$$

$$(y - y_0) \sin \genfrac{}{}{0pt}{}{y}{x} = x' \sin \genfrac{}{}{0pt}{}{x'}{x} + y' \sin \genfrac{}{}{0pt}{}{y'}{x}$$

transform from oblique axes to oblique.

2. The equations

$$x - x_0 = x' \cos \genfrac{}{}{0pt}{}{x'}{x} + y' \cos \genfrac{}{}{0pt}{}{y'}{x}$$

$$y - y_0 = x' \sin \genfrac{}{}{0pt}{}{x'}{x} + y' \sin \genfrac{}{}{0pt}{}{y'}{x}$$

transform from rectangular to oblique axes.

3. The equations

$$x - x_0 = x' + y' \cos \tfrac{y'}{x}$$

$$y - y_0 = y' \sin \tfrac{y'}{x}$$

transform from rectangular to oblique axes when x is parallel to x'.

4. The equations

$$x - x_0 = x' \cos \tfrac{x'}{x}$$

$$y - y_0 = y' + x' \sin \tfrac{x'}{x}$$

transform from rectangular to oblique axes when y is parallel to y'.

5. The equations

$$(x - x_0) \sin \tfrac{x}{y} = x' \sin \tfrac{x'}{y} + y' \cos \tfrac{x'}{y}$$

$$(y - y_0) \sin \tfrac{y}{x} = x' \sin \tfrac{x'}{x} + y' \cos \tfrac{x'}{x}$$

transform from oblique axes to rectangular.

6. The equations

$$(x - x_0) \sin \tfrac{x}{y} = x' \sin \tfrac{x}{y} + y' \cos \tfrac{x}{y}$$

$$(y - y_0) \sin \tfrac{y}{x} = y'$$

transform from oblique axes to rectangular when x is parallel to x'.

7. *The equations*

$$(x-x_0)\sin\genfrac{}{}{0pt}{}{x}{y}=x'$$

$$(y-y_0)\sin\genfrac{}{}{0pt}{}{y}{x}=x'\sin\genfrac{}{}{0pt}{}{x'}{x}+y'\cos\genfrac{}{}{0pt}{}{x'}{x}$$

transform from oblique axes to rectangular when y *is parallel to* y'.

8. *The equations*

$$x-x_0=x'\cos\genfrac{}{}{0pt}{}{x'}{x}-y'\sin\genfrac{}{}{0pt}{}{x'}{x}$$

$$y-y_0=x'\sin\genfrac{}{}{0pt}{}{x'}{x}+y'\cos\genfrac{}{}{0pt}{}{x'}{x}$$

transform from rectangular axes to rectangular.

9. *The equations*

$$(x-x_0)\sin\genfrac{}{}{0pt}{}{x}{y}=\rho\sin\genfrac{}{}{0pt}{}{\rho}{y}$$

$$(y-y_0)\sin\genfrac{}{}{0pt}{}{y}{x}=\rho\sin\genfrac{}{}{0pt}{}{\rho}{x}$$

transform from oblique to polar co-ordinates.

10. *The equations*

$$x-x_0=\rho\cos\genfrac{}{}{0pt}{}{\rho}{x}$$

$$y-y_0=\rho\sin\genfrac{}{}{0pt}{}{\rho}{x}$$

$$\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}$$

transform from rectangular to polar co-ordinates.

CHAPTER III.

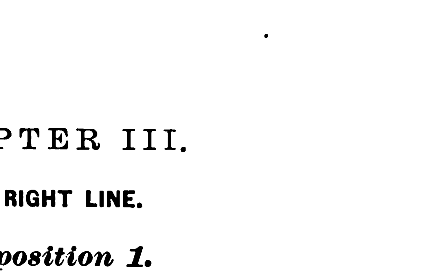
THE RIGHT LINE.

Proposition 1.

90. ***Theorem.***—*The equation*

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

represents a right line through two given points; in which x ***and*** y ***are the co-ordinates of any point of the line, and*** x_1 ***and*** y_1, x_2 ***and*** y_2 ***are those of the given points.***



For, let P be any point of the line through P_1 and P_2, the given points. Then from similar triangles $PQ : P_1Q_1 :: P_1Q : P_2Q_1$

$$\therefore \frac{PQ}{P_1Q_1}=\frac{P_1Q}{P_2Q_1}$$

$$\therefore \qquad \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}, \text{ or } \frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

This right line is conceived of, as indefinitely extended in either direction, and is called the *locus* of P. It may be drawn across any angle, first, second, third or fourth, according to the position of P_1 and P_2.

91. ***Schol. 1.***—The equation $\frac{y_3-y_1}{x_3-x_1}=\frac{y_2-y_1}{x_2-x_1}$ expresses the relation that must hold in order that some point (x_3, y_3), (*i. e.*, P_3), shall be upon this line.

For evidently P_3 must coincide with some one of the infinite number of positions of P. Clear of fractions and we have,

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0,$$

which, by Art. 31, is the relation which holds when a triangle reduces to a straight line. This is called the *equation of condition* that three points shall be upon one right line.

92. The distances OA and OB are called **intercepts,** being the parts of the axes between the origin and the line. It will be convenient to use a and b to denote intercepts on x and y respectively.

93. ***Examples*** in either rectangular or oblique co-ordinates.

(1.) What is the equation of the line through the points $(3, 2)$ and $(2, -4)$? *Ans.* $y = x - 19$.

(2.) Through the points $(2, -3)$ and $(-4, 1)$?

Ans. $3y = -2x - 5$.

(3.) Are the points $(2, 3)$, $(1, -1)$ and $(-1, -9)$ on the same straight line?

(4.) The points $(3, 4)$, $(1, -1)$ and $(-3, -5)$?

(5.) Write the equations of the lines through the points in ex. (3), and in ex. (4).

(6.) Draw the lines whose equations are obtained in the examples of this article.

94. ***Exercise.***—Show that the form of the equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

is not changed by moving the origin to any point (x_0, y_0).

Proposition 2.

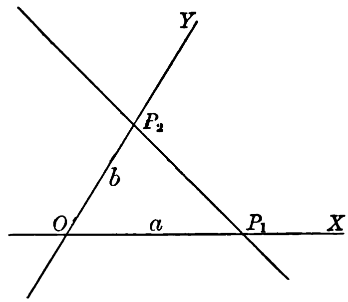
95. ***Theorem.***—*The equation*

$$\frac{x}{a} + \frac{y}{b} = 1$$

represents a right line; in which x *and* y *are the co-ordi-*

nates of any point of the line, and** a **and** b **are the intercepts.

For, in Art. 90 let P_1 fall on the axis of x, and P_2 on the axis of y.



$\therefore \quad x_1 = OP_1 = a$, and $y_1 = 0$

$x_2 = 0$, and $y_2 = OP_2 = b$

$\therefore$ we have $\dfrac{y-0}{x-a} = \dfrac{b-0}{0-a}$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1.$$

This equation is symmetrical, and of the *zero* degree when we consider both x and y and also the *constants* a and b. In x and y only, it is of the *first* degree.

96. When a and b are *given*, we have the equation of a *particular* line. But a and b may represent any intercepts, and in this sense the equation is said to be the *general equation* of a right line in terms of its intercepts.

97. ***Examples.***—Reduce the equations of the straight lines obtained in Art. 93 to the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

i. e., so that the right hand member is $+1$:

e. g., if $\quad 3y = -x - 7$

then, transposing and dividing,

$$\frac{x}{-7} + \frac{y}{-\frac{7}{3}} = 1$$

$\therefore$ the intercepts are -7, and $\frac{-7}{3}$.

Also show that when (Art. 90)

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}, \text{ then } a = \frac{x_1y_2 - x_2y_1}{y_2-y_1}, \text{ and } b = \frac{y_1x_2 - y_2x_1}{x_2-x_1}.$$

Proposition 3.

98. ***Theorem.***—*The equation*

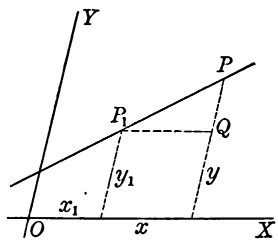
$$\frac{y-y_1}{x-x_1}=m$$

represents a right line through one given point; in which x and y are the co-ordinates of any point of the line, x_1 and y_1 those of the given point, and $m=\dfrac{\sin {l \atop x}}{\sin {y \atop l}}$.

For, from the triangle PQP_1 by trig.

$y-y_1 : x-x_1 :: \sin QP_1P : \sin P_1PQ$

$$\therefore \quad \frac{y-y_1}{x-x_1}=\frac{\sin {l \atop x}}{\sin {y \atop l}}=m,$$



in which l denotes the direction of the line—*i. e.*, $\sin {l \atop x}$ is read, "sine of the angle between the axis of x and the line l."

99. ***Cor. 1.***—When ${y \atop x}=\omega=90°$, then $m=\dfrac{\sin {l \atop x}}{\cos {l \atop x}}=\tan {l \atop x}$;

$\therefore$ in *rectangulars* $m=$ the tangent of the angle which the line makes with the axes of x.

100. ***Cor. 2.***—If $\dfrac{y-y_1}{x-x_1}=\dfrac{\sin x}{\cos {l \atop x}}=\tan {l \atop x}$,

we have, $$\frac{y-y_1}{\sin {l \atop x}}=\frac{x-x_1}{\cos {l \atop x}}=l,$$

in which $l=PP_1$ is the distance of any point P from the given point P_1, and is measured along the line.

This equation may also be written

$$(x-x_1)\sin {l \atop x}-(y-y_1)\cos {l \atop x}=0.$$

101. ***Schol. 1.***—It is to be noticed that when the value of m is *given* in the equation

$$\frac{y-y_1}{x-x_1}=m$$

a particular line is represented; when it is not given, the equation may represent *any one* of the lines through P_1, and in this sense it is said to be the *general* equation of the line through one given point.

102. ***Schol. 2.***—In Art. 90 suppose that P_2 coincides with P_1,

then, $\quad \dfrac{y-y_1}{x-x_1}=\dfrac{0}{0}=m=$ some indeterminate quantity.

This algebraic indetermination expresses the known fact that an infinite number of different lines can be drawn through P_1.

Proposition 4.

103. ***Theorem.***—*The equation*

$$y-b=mx$$

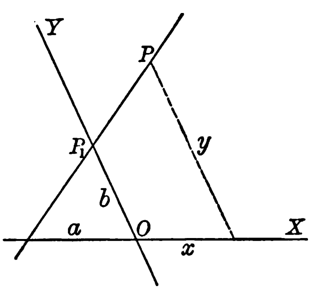
represents a right line; in which x *and* y *are the co-ordinates of any point of the line,* b *is the intercept on* y*,*

and $m=\dfrac{\sin \frac{l}{x}}{\sin \frac{y}{l}}$.

For, in Art. 98 let P_1 fall upon the axis of y.

$\therefore \quad x_1=0$ and $y_1=b$,

$\therefore \quad \dfrac{y-b}{x}=m,\quad$ or $y-b=mx$.



Similarly if P_1 be upon the axis of x,
then $\quad x_1=a,\;$ and $y_1=0$.

$\therefore \quad \dfrac{y}{x-a}=m,\quad$ or $x-a=\dfrac{y}{m}$.

104. *Cor. 1.*—When $\overset{y}{x} = 90°$, then $m = \tan \overset{l}{x}$, or $\frac{1}{m} = \cot \overset{l}{x}$

The equation $y - b = mx$ might therefore be written,

$$y - b = x \tan \overset{l}{x}, \quad \text{or } y = x \tan \overset{l}{x} + b.$$

105. *Cor. 2.*—If $b = 0$, $y = mx$ is the equation of a right line through the *origin*—*i. e.*, if the equation *contains no constant term, then the origin is on the line.*

If $y = x$, the line *bisects* the angle $\overset{y}{x}$, and passes through the origin, and $y = x + b$ is *parallel* to this bisecting line.

106. *Cor. 3.*—When $m = 0$, then $y = b$, and the line is *parallel* to the axes of x.

When also $b = 0$, then $y = 0$, and the line *coincides* with the axis of x.

When in the last equation of Art. 103, $\frac{1}{m} = 0$, then $x = a$, and the line is *parallel* to the axis of y.

When also $a = 0$, then $x = 0$, and the line *coincides* with the axis of y.

107. *Examples.*—(1.) Determine the intercepts of the right line through the point $(1, 5)$, when $m = \frac{3}{4}$. *Ans.* $a = -5\frac{2}{3}$, $b = 4\frac{1}{4}$.

(2.) Determine the intercepts of the right line through the point $(4, -5)$, when $m = \frac{3}{2}$. *Ans.* $a = 7\frac{1}{3}$, $b = -11$.

(3.) Find the equation and intercepts of a right line through the point $(-2, -4)$, and perpendicular to the axis of y, when $\omega = \frac{\pi}{3}$.

Ans. Equation is $y = -\frac{x}{2} - 5$. Intercepts, $a = -10$, $b = -5$.

(4.) Find the values of m in the examples of Article 93.

(5.) Show that the intercept $a = \frac{-b}{m}$.

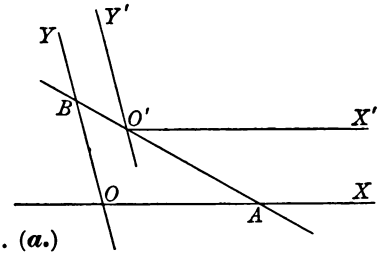
108. *Exercise.*—Derive the equation of Art. 103 from the figure, and prove the equations of Arts. 90 and 98 from it, when $\omega = 90°$.

Proposition 5.

109. ***Theorem.***—*The general equation of the first degree*

$$Ax + By + C = 0$$

represents some right line; in which x and y are the co-ordinates of any point of the line, and A, B and C may each have any real value whatever.



For if the origin be moved to a parallel position by the equations of (Art. 23), viz.: $x = x_0 + x'$, and $y = y_0 + y'$, we obtain by substitution,

$A\,(x_0 + x') + B\,(y_0 + y') + C = 0$;

or $Ax' + By' + Ax_0 + By_0 + C = 0$ . . . (*a.*)

Values for x_0 and y_0 have not yet been assigned, and we may evidently give them whatever values we please. Let them have such values that $\qquad Ax_0 + By_0 + C = 0$. . . . (*b.*)

There can be an infinite number of such values—*i. e.*, of positions of the new origin (x_0, y_0),—for if A, B and C are given, and we assign any value whatever to y_0, we can evidently find from eq. (*b.*) a corresponding value of x_0 such as will verify the equation. Let the new origin (x_0, y_0) be at any point O', that satisfies eq. (*b.*); then eq. (*a.*) becomes $\qquad Ax' + By' = 0$. . . . . (*c.*)

Equation (*c.*) then represents the same thing referred to new axes that (*a.*) represented when referred to the primitive axes.

If in (*c.*) $x' = 0$, then $y' = 0$, which shows that this new origin is a point on the line, straight or curved, which is represented by (*a.*) and (*c.*).

We may omit the primes and write eq. (*c.*), $Ax + By = 0$, for the primes are used only to distinguish conveniently one system of axes from another.

Now multiply by $\sin\frac{y}{x}$,

$$\therefore\quad Ax \sin\frac{y}{x} + By \sin\frac{y}{x} = 0,$$

or (Art. 13), $\qquad Ax \sin\frac{x}{y} - By \sin\frac{y}{x} = 0.$

Next change the direction of the axes. Substituting from Art. 69,

$$\therefore \quad A\left(x' \sin \genfrac{}{}{0pt}{}{x'}{y} + y' \sin \genfrac{}{}{0pt}{}{y'}{y}\right) - B\left(x' \sin \genfrac{}{}{0pt}{}{x'}{x} + y' \sin \genfrac{}{}{0pt}{}{y'}{x}\right) = 0.$$

Rearranging the terms we obtain,

$$\left(A \sin \genfrac{}{}{0pt}{}{x'}{y} - B \sin \genfrac{}{}{0pt}{}{x'}{x}\right) x' + \left(A \sin \genfrac{}{}{0pt}{}{y'}{y} - B \sin \genfrac{}{}{0pt}{}{y'}{x}\right) y' = 0 \ldots (d.)$$

Since the angles $\genfrac{}{}{0pt}{}{x'}{y}$, $\genfrac{}{}{0pt}{}{x'}{x}$, $\genfrac{}{}{0pt}{}{y'}{y}$, $\genfrac{}{}{0pt}{}{y'}{x}$, are not yet determined, and two of them, either $\genfrac{}{}{0pt}{}{x'}{y}$ and $\genfrac{}{}{0pt}{}{x'}{x}$, or $\genfrac{}{}{0pt}{}{y'}{y}$ and $\genfrac{}{}{0pt}{}{y'}{x}$, or $\genfrac{}{}{0pt}{}{x'}{y}$ and $\genfrac{}{}{0pt}{}{y'}{x}$ or $\genfrac{}{}{0pt}{}{x'}{x}$ and $\genfrac{}{}{0pt}{}{y'}{y}$, are independent, we may assign any values we please to the two, or affix such conditions as will determine their values. Let the two following conditions hold:

$$\text{1st. } A \sin \genfrac{}{}{0pt}{}{x'}{y} - B \sin \genfrac{}{}{0pt}{}{x'}{x} = 0, \quad \text{and 2d. } A \sin \genfrac{}{}{0pt}{}{y'}{y} - B \sin \genfrac{}{}{0pt}{}{y'}{x} \lesseqgtr 0.$$

Substituting the 1st condition in equation (*d.*), we have,

$$\left(A \sin \genfrac{}{}{0pt}{}{y'}{y} - B \sin \genfrac{}{}{0pt}{}{y'}{x}\right) y' = 0. \qquad \therefore \quad y' = 0,$$

by the 2d condition. From which we see that there is no point represented by this equation which does not coincide with the axis of x', for $y' = 0$ is evidently the equation of a line coinciding with the axis of x', but the axis of x' is a right line. $\therefore$ (*a.*) represents a right line.

110. *Cor. 1.*—To reduce the general form of the equation of a right line,

$$Ax + By + C = 0,$$

to the form of the equation in terms of the intercepts (Art. 95), transpose and divide by $-C$.

$$\therefore \quad \frac{Ax}{-C} + \frac{By}{-C} = 1, \quad \text{or,} \quad \frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}} = 1$$

is the form; in which

$$a = -\frac{C}{A}, \text{ and } b = -\frac{C}{B}$$

are the intercepts.

111. ***Cor. 2.***—To reduce the general form to the form of one intercept (Art. 103), solve with reference to y,

$$\therefore \quad y = -\frac{A}{B}x - \frac{C}{B};$$

in which, as before, $\quad b = -\frac{C}{B}, \quad$ and $\quad -\frac{A}{B} = \frac{\sin \frac{l}{x}}{\sin \frac{l}{y}}$.

112. ***Schol. 1.***—The equation

$$(A_1 x + B_1 y + C_1)(A_2 x + B_2 y + C_2) = 0$$

represents two lines separately, for by the general theory of equations each factor $= 0$.

113. ***Schol. 2.***—If $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ are the equations of the lines (1) and (2), then $A_1 x_i + B_1 y_i + C_1 = 0$ and $A_2 x_i + B_2 y_i + C_2 = 0$ hold respecting the co-ordinates of the point of intersection, (x_i, y_i). Eliminating, we have,

$$x_i = -\frac{C_1 B_2 - C_2 B_1}{A_1 B_2 - A_2 B_1} \quad \text{and } y_i = -\frac{C_1 A_2 - C_2 A_1}{B_1 A_2 - B_2 A_1}.$$

114. ***Schol. 3.***—The general equation of the first degree, viz.,

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0,$$

contains but *two* arbitrary constants.

N. B.—We shall, for convenience, speak of "the line $Ax + By + C = 0$," meaning the line which the equation $Ax + By + C = 0$ represents.

Proposition 6.

115. ***Theorem.***—*The equation*

$$k_1 (A_1 x + B_1 y + C_1) + k_2 (A_2 x + B_2 y + C_2) = 0$$

represents some right line passing through the intersection of the lines $A_1 x + B_1 y + C_1 = 0$ *and* $A_2 x + B_2 y + C_2 = 0$; *in which* k_1 *and* k_2 *are any multipliers.*

For, by Art. 109 it is the equation of some right line, since it may be reduced to the form,

$$(A_1k_1 + A_2k_2)\,x + (B_1k_1 + B_2k_2)\,y + C_1k_1 + C_2k_2 = 0,$$

or, as it may be written, $A_3x + B_3y + C_3 = 0.$

Moreover, the equation is evidently satisfied when both

$$A_1x + B_1y + C_1 = 0, \text{ and } A_2x + B_2y + C_2 = 0,$$

provided the values of x and y are the same—*i. e.*, *simultaneous*, in the two expressions. But x and y can have the same values in lines (1) and (2) only at their intersection; therefore the line (3) passes through the intersection of lines (1) and (2).

116. ***Cor. 1.***—The equation of line (1) is, $-k_1(A_1x + B_1y + C) = 0.$ The equation of line (2) is, $-k_2(A_2x + B_2y + C_2) = 0$, and that of line (3) is, $k_1(A_1x + B_1y + C_1) + k_2(A_2x + B_2y + C_2) = 0.$ Add these together, and they vanish *identically*. Therefore, when the equations of three lines, after being each multiplied through by any constants k_1, k_2 and k_3, can be added so as to vanish *identically*, the lines pass through one common point, and are called *convergents*, or a pencil of three rays.

117. ***Cor. 2.***—If lines (1), (2) and (3), whose equations are,

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0, \quad A_3x + B_3y + C_3 = 0,$$

intersect in a point, by elimination we obtain,

$$C_1(A_2B_3 - A_3B_2) + C_2(A_3B_1 - A_1B_3) + C_3(A_1B_2 - A_2B_1) = 0,$$

which is the equation of condition that three lines intersect in a point.

118. ***Schol.***—When the constants of lines (1) and (2) have definite numerical values, then (1) and (2) are determinate lines. But in combining (1) and (2) to obtain (3) (Art. 115), it is still possible to assign at pleasure any values to k_1 and k_2, thus producing any one of an infinite number of lines (3), each of which passes through the intersection of (1) and (2), and each of which satisfies the equation of condition in Art. 117.

E. G., if line (1) is $x + y + 2 = 0$, and line (2) is $x - 2y - 1 = 0$, then line (3) is, $2x - y + 1 = 0$, when $k_1 : k_2 = 1$, and is, $3x + 3 = 0$, when $k_1 : k_2 = 2$, etc., etc.

119. *Exercise.*—Prove that the three lines (1), (2) and (3) enclose the triangle whose area is t, when

$$\frac{[C_1(A_2B_3-A_3B_2)+C_2(A_3B_1-A_1B_3)+C_3(A_1B_2-A_2B_1)]^2}{(A_2B_3-A_3B_2)(A_3B_1-A_1B_3)(A_1B_2-A_2B_1)}=\pm 2t$$

by substituting $\quad x_1=\dfrac{C_1A_2-C_2A_1}{A_1B_2-A_2B_1},\quad y_1=\dfrac{C_1B_2-C_2B_1}{A_1B_2-A_2B_1}$, etc.,

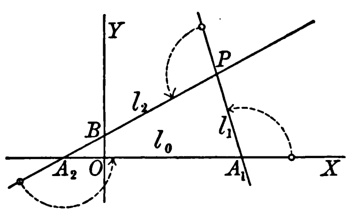
in the formula of Art. 31. Also prove Art. 117 from the above.

Proposition 7.

120. *Theorem.*—*The equation*

$$\tan\frac{l_1}{l_0}\tan\frac{l_2}{l_1}\tan\frac{l_0}{l_2}=\tan\frac{l_1}{l_0}+\tan\frac{l_2}{l_1}+\tan\frac{l_0}{l_2}$$

expresses the relations which subsist between the angles of any plane triangle in which l_0, l_1 and l_2 are the directions of its sides. (Art. 12.)



For (Art. 12), $360^\circ=\frac{l_1}{l_0}+\frac{l_2}{l_1}+\frac{l_0}{l_2}$

this being the sum of the external angles of the triangle.

$$\therefore\quad 360^\circ-\frac{l_2}{l_1}=\frac{l_1}{l_0}+\frac{l_0}{l_2},$$

and by trig. $\quad \tan\left(360^\circ-\frac{l_2}{l_1}\right)=\dfrac{\tan\frac{l_1}{l_0}+\tan\frac{l_0}{l_2}}{1-\tan\frac{l_1}{l_0}\tan\frac{l_0}{l_2}}=-\tan\frac{l_2}{l_1}.$

Clear of fractions, $\therefore\ \tan\frac{l_1}{l_0}\tan\frac{l_2}{l_1}\tan\frac{l_0}{l_2}=\tan\frac{l_1}{l_0}+\tan\frac{l_2}{l_1}+\tan\frac{l_0}{l_2}.$

Proposition 8.

121. *Theorem.*—*The equation*

$$m_0=\frac{m_1-m_2}{1+m_1m_2}$$

expresses the value of m_0, the tangent of the angle between any two lines $y=m_1x+b_1$, and $y=m_2x+b_2$.

For, let the axis of x,—*i. e.*, the line $y=0$, having the direction l_0, together with the line $y=m_1x+b_1$, having the direction l_1, and the line $y=m_2x+b_2$, having the direction l_2, form a triangle, then (Art. 120), $m_0=\tan{}^{l_1}_{l_2}$, $m_1=\tan{}^{l_1}_{x}$, $m_2=\tan{}^{l_2}_{x}$.

By Art. 120, $$-\tan{}^{l_2}_{l_1}=\frac{\tan{}^{l_1}_{x}+\tan{}^{x}_{l_2}}{1-\tan{}^{l_1}_{x}\tan{}^{x}_{l_2}}$$

$$\therefore \quad \text{(Art. 13)} \quad \tan{}^{l_1}_{l_2}=\frac{\tan{}^{l_1}_{x}-\tan{}^{l_2}_{x}}{1+\tan{}^{l_1}_{x}\tan{}^{l_2}_{x}}.$$

$$\therefore \quad m_0=\frac{m_1-m_2}{1+m_1m_2}. \ldots (e.)$$

$$\text{Similarly,} \quad m_2=\frac{m_1-m_0}{1+m_1m_0} \ldots (f.)$$

122. ***Cor. 1.***—If ${}^{l_1}_{l_2}=0$, then $m_0=0$.

$\therefore$ (*e.*) reduces to $m_1-m_2=0$.

But (Art. 111) $$m_1=\frac{A_1}{B_1}, \text{ and } m_2=\frac{A_2}{B_2}$$

$$\therefore \quad \frac{A_1}{B_1}-\frac{A_2}{B_2}=0, \text{ or } A_1B_2-A_2B_1=0.$$

Hence the condition of parallelism between the two lines is,

$$m_1-m_2=0, \text{ or } \frac{A_1}{B_1}-\frac{A_2}{B_2}=0,$$

as may also be seen from Art. 113.

123. ***Cor. 2.***—If ${}^{l_1}_{l_2}=90^\circ$, then $m_0=\infty$.

$\therefore$ (*e.*) reduces to $1+m_1m_2=0$, or $m_1=-\dfrac{1}{m_2}$,

$\therefore$ (Art. 111) $A_1A_2+B_1B_2=0$.

Either equation is the condition of perpendicularity of two lines.

124. ***Cor. 3.***—If ${}^{l_2}_{x}=0$, then $m_2=0$, or (Art. 111) $\dfrac{-A_2}{B_2}=0$;

$\therefore$ $A_2 = 0$, is the condition of parallelism to axis of x,—*i. e.*, the line $B_2 y + C_2 = 0$ is parallel to the axis of x.

If $\frac{l_2}{x} = 90°$, then $m_2 = \infty$, or (Art. 111) $\frac{-A_2}{B_2} = \infty$;

$\therefore$ $B = 0$ is the condition of perpendicularity to the axis of x,—*i. e.*, the line $A_2 x + C_2 = 0$ is perpendicular to the axis of x.

125. ***Cor. 4.***—Since $\tan \frac{l_1}{l_2} = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{A_1 B_2 - A_2 B_1}{A_1 A_2 + B_1 B_2}$, (Art. 111),

by trig., $\sin \frac{l_1}{l_2} = \frac{m_1 - m_2}{\sqrt{(1 + m_1^2)}\sqrt{(1 + m_2^2)}} = \frac{A_1 B_2 - A_2 B_1}{\sqrt{(A_1^2 + B_1^2)}\sqrt{(A_2^2 + B_2^2)}}$

and $\cos \frac{l_1}{l_2} = \frac{1 + m_1 m_2}{\sqrt{(1 + m_1^2)}\sqrt{(1 + m_2^2)}} = \frac{A_1 A_2 + B_1 B_2}{\sqrt{(A_1^2 + B_1^2)}\sqrt{(A_2^2 + B_2^2)}}$.

126. ***Examples.***—(1.) Given the equation

$$2(3x - y + 4) - 5(2x + 3y - 1) = 0;$$

to construct the three intersecting lines indicated by the equation, and find the co-ordinates of their point of intersection.

Ans. $x_i = -1$, $y_i = 1$.

(2.) Show numerically, by means of Art. 121 and the equations

$$y = 3x + 5, \quad 4y = x + 8 \quad \text{and} \quad y = -x - 1,$$

that the relation of Art. 120 holds.

127. ***Exercise.***—Prove that when the axes are oblique,

$$\tan \frac{l_1}{l_2} = \frac{(A_1 B_2 - A_2 B_1) \sin \frac{y}{x}}{A_1 A_2 + B_1 B_2 - (A_1 B_2 + A_2 B_1) \cos \frac{y}{x}}.$$

Proposition 9.

128. ***Theorem.***—(*Rectangular co-ordinates.*)

The line $y = m_1 x + b_2$ ***is parallel to*** $y = m_1 x + b_1$ ***by Art. 122.***

The line $y - y_1 = m_1(x - x_1)$ ***passes through*** (x_1, y_1), ***and is also parallel to*** $y = m_1 x + b_2$.

The line $A_1x+B_1y+C_2=0$ *is parallel to* $A_1x+B_1y+C_1=0$ *by Art. 122.*

The line $A_1(x-x_1)+B_1(y-y_1)=0$ *passes through* (x_1, y_1), *and is parallel to* $A_1x+B_1y+C_1=0$.

The line $y=\dfrac{-1}{m_1}x+b_2$ *is perpendicular to* $y=m_1x+b_1$ *by Art. 123.*

The line $y-y_1=\dfrac{-1}{m_1}(x-x_1)$ *is perpendicular to* $y=m_1x+b$ *and passes through* (x_1, y_1). *The same line is also perpendicular to* $y-y_1=m_1(x-x_1)$ *at the point* (x_1, y_1).

The line $B_1x-A_1y+C_2=0$ *is perpendicular to* A_1x+B_1y $C_1=0$ *by Art. 123.*

The line $B_1(x-x_1)-A_1(y-y_1)=0$ *is perpendicular to* $A_1x+B_1y+C_1=0$, *and passes through* (x_1, y_1). *It is also perpendicular to* $A_1(x-x_1)+B_1(y-y_1)=0$, *at the point* (x_1, y_1).

129. ***Example.***—Find the equations of two lines at right angles with each other, the first of which passes through the point $(-2, 5)$, and has $\tan\frac{l_1}{x}=2$, while the second passes through the point $(4, 1)$.

Ans. $y=2x+9$ and $2y+x=6$.

130. ***Exercises.***—(*Oblique co-ordinates.*) Show that

(1.) $y=-\dfrac{1+m_1\cos\frac{y}{x}}{m_1+\cos\frac{y}{x}}x+b_2$, and $y=m_1x+b_1$ are perpendicular.

(2.)
$$(A_1\cos\tfrac{y}{x}-B_1)x-(B_1\cos\tfrac{y}{x}-A_1)y+C_2=0$$
is perpendicular to $A_1x+B_1y+C_1=0$.

(3.)
$$y-y_1=\frac{1+m_1\cos\frac{y}{x}}{m_1+\cos\frac{y}{x}}(x-x_1)$$
passes through (x_1, y_1), and is perpendicular to $y=m_1x+b_1$.

(4.) $(A_1\cos\frac{y}{x}-B_1)(x-x_1)-(B_1\cos\frac{y}{x}-A_1)(y-y_1)=0$ passes through (x_1, y_1), and is perpendicular to $A_1x+B_1y+C_1=0$.

Proposition 10.†

131. ***Theorem.***—(*Rectangular co-ordinates.*) *The equations*

$$y = m_2 x + b_2, \quad \text{and} \quad y - y_1 = m_2 (x - x_2),$$

which may by Art. 121 be written

$$y = \frac{m_1 - m_0}{1 + m_1 m_0} x + b_2, \quad \text{and} \quad y - y_1 = \frac{m_1 - m_0}{1 + m_1 m_0} (x - x_1),$$

represent lines making an angle with any given line

$$y = m_1 x + b_1, \quad \text{or} \quad y - y_1 = m_1 (x - x_1),$$

such that the tangent of this angle is $m_0 = \tan \frac{l_1}{l_2}$.

This is evident from Articles 103 and 121.

It is also to be noticed that two lines can be drawn each making an angle of the same number of degrees with $y = m_1 x + b_1$, but one makes a positive and the other a negative angle, corresponding to $+m_0$ and $-m_0$ respectively, from which we have two values of m_2.

132. ***Schol. 1.***—If $m_1 = 0$, the given line is parallel to the axis of x, $\therefore$ $y = \mp m_0 x + b_2$ and $y - y_1 = \mp m_0 (x - x_1)$ are the equations.

133. ***Schol. 2.***—If $m_1 = \infty$, the given line is perpendicular to the axis of x. Since in this case

$$\frac{m_1 - m_0}{1 + m_1 m_0} = \frac{1 - \dfrac{m_0}{m_1}}{\dfrac{1}{m_1} + m_0} = \frac{1}{m_0},$$

$$\therefore \quad y = \pm \frac{1}{m_0} x + b_2, \quad \text{and} \quad y - y_1 = \pm \frac{1}{m_0} (x - x_1)$$

are the equations.

134. ***Schol. 3.***—If $m_1 = m_0$, then $y = b_2$ and $y - y_1 = 0$ are the equations. But if $m_1 = -m_0$ the equations become

$$y = \frac{2m_0}{1 - m_0^2} x + b_2, \quad \text{and} \quad y - y_1 = \frac{2m_0}{1 - m_0^2} (x - x_1);$$

or $$y = x \tan 2 \left(\frac{l_1}{l_2}\right) + b_2, \quad \text{and} \quad y - y_1 = (x - x_1) \tan 2 \left(\frac{l_1}{l_2}\right).$$

135. ***Schol. 4.***—If $m_1 = -\frac{1}{m_0}$, then $x = 0$ and $x - x_1 = 0$ are the equations. But if $m_1 = \frac{1}{m_0}$, then the equations become

$$y = x \cot 2\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right) + b_2 \text{ and } y - y_1 = (x - x_1) \cot 2\left(\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right).$$

136. ***Schol. 5.***—If $m_0 = 0$, the line is parallel to the given line, and the equations become $y = m_1 x + b_2$, and $y - y_1 = m_1(x - x_1)$.

137. ***Schol. 6.***—If $m_0 = \infty$, then $\dfrac{m_1 - m_0}{1 + m_1 m_0} = \dfrac{\dfrac{m_1}{m_0} - 1}{\dfrac{1}{m_0} + m_1} = -\dfrac{1}{m_1}$,

$\therefore\ y = -\frac{1}{m_1}x + b_2$, **and** $y - y_1 = -\frac{1}{m_1}(x - x_1)$ are the equations, and the line is perpendicular to the given line.

138. ***Examples.***—(1.) Form the equation of a line making an angle of $30°$ with the line $7y - x\sqrt{3} + 2 = 0$, and having an intercept on y of -4.

Ans. $y = -\frac{5}{17}\sqrt{3}.x - 4$, or $y = \frac{9}{17}\sqrt{3}.x - 4$.

(2.) Form the equations of two lines making with each other an angle of $45°$, the first passing through the two points $(1, 2)$ and $(-4, -3)$, and the second through the point $(1, -3)$.

Ans. $y = x + 1$, and $y = -3$, or $x = 1$.

Proposition 11.

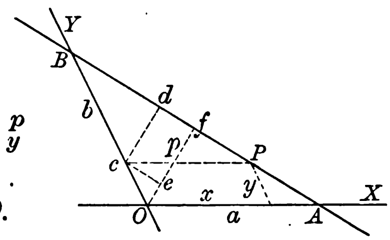
139. ***Theorem.***—*The equation*

$$x \cos \tfrac{p}{x} + y \cos \tfrac{p}{y} - p = 0$$

represents a right line; in which p *is the length of the perpendicular let fall from the origin upon the line, and* $\frac{p}{x}$ *and* $\frac{p}{y}$ *are the angles between the co-ordinate axes and the perpendicular.*

Let $\frac{x}{a}+\frac{y}{b}-1=0$ represent AB (Art. 95). Multiply by p,

$$\therefore\quad \frac{p}{a}x+\frac{p}{b}y-p=0.$$



By trig. $\frac{p}{a}=\cos\frac{p}{x}$, and $\frac{p}{b}=\cos\frac{p}{y}$

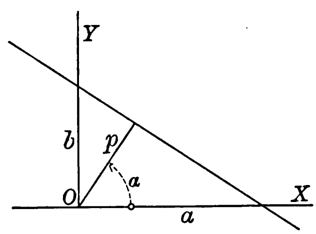
$$\therefore\quad x\cos\frac{p}{x}+y\cos\frac{p}{y}-p=0.$$

This is the equation of a right line in terms of the *direction cosines* of its *perpendicular*, and p is always considered as *positive.*

The equation may also be derived directly from the figure,

$$Oe+cd=Oc.\cos eOc+cP.\cos dcP=y\cos\frac{y}{p}+x\cos\frac{x}{p}=p.$$

140. ***Cor. 1.***—If $\frac{y}{x}=90°$,



$$\frac{p}{y}=\frac{x}{y}+\frac{p}{x}=\frac{p}{x}-\frac{y}{x}=\frac{p}{x}-90°.$$

$$\therefore\quad \cos\frac{p}{y}=\cos\left(\frac{p}{x}-90°\right)=\sin\frac{p}{x}.$$

$$\therefore\quad x\cos\frac{p}{x}+y\sin\frac{p}{x}-p=0,$$

$$\text{or, if } \alpha=\frac{p}{x},\quad x\cos\alpha+y\sin\alpha-p=0.$$

141. ***Cor. 2.***—The value of p, from Art. 139, is $p=a\cos\frac{p}{x}=a\sin\frac{l}{x}$. But by trig. we have from the triangle AOB

$$\sin\frac{l}{x}=\frac{b\sin\frac{y}{x}}{AB}=\frac{b\sin\frac{y}{x}}{\sqrt{\left(a^2+b^2-2ab\cos\frac{y}{x}\right)}}.$$

$$\therefore\quad p=\frac{ab\sin\frac{y}{x}}{\sqrt{\left(a^2+b^2-2ab\cos\frac{y}{x}\right)}}.$$

142. ***Cor. 3.***—To reduce the form $Ax + By + C = 0$ to the form $x \cos\frac{p}{x} + y \sin\frac{p}{x} - p = 0$, let $\frac{y}{x} = 90°$ in the formula of the preceding article; then $p = \frac{ab}{\sqrt{(a^2 + b^2)}}$. By Art. 110, $a = -\frac{C}{A}$, $b = -\frac{C}{B}$.

$$\therefore\ p = \frac{\frac{C^2}{AB}}{\sqrt{\left(\frac{C^2}{A^2} + \frac{C^2}{B^2}\right)}} = \frac{\frac{C^2}{AB}}{C\sqrt{\left(\frac{A^2 + B^2}{A^2 B^2}\right)}} = \frac{C}{\sqrt{(A^2 + B^2)}}$$

$$\cos\frac{p}{x} = \frac{p}{c} = \frac{-A}{\sqrt{(A^2 + B^2)}}. \qquad \sin\frac{p}{x} = \frac{p}{b} = \frac{-B}{\sqrt{(A^2 + B^2)}}.$$

$$\therefore \text{ the form is } \quad \frac{-A}{\sqrt{(A^2 + B^2)}}\, x + \frac{-B}{\sqrt{(A^2 + B^2)}}\, y = \frac{C}{\sqrt{(A^2 + B^2)}}.$$

Hence to perform the reduction in any case, divide through by $\sqrt{(A^2 + B^2)}$. It is to be noticed that if the reduction be applied to itself, the divisor is of the form $\sqrt{(\sin^2 a + \cos^2 a)} = 1$, which does not change the form.

143. ***Example.***—Form the equations, in terms of the perpendicular from the origin and its direction cosines, of the diagonals of a parallelogram, each of whose sides is 4, and one of whose angles is $60°$, taking two adjacent sides as axes of reference

$$Ans.\ \tfrac{1}{2} x\sqrt{3} + \tfrac{1}{2} y\sqrt{3} - 3.464 = 0, \text{ or } x + y = 4,$$

$$\text{and } x \cos 60° - y \cos 60° = 0, \quad \text{or } x = y.$$

144. ***Exercise.***—Prove that with oblique axes the divisor corresponding to that in Art. 142 is

$$\frac{\sqrt{\left(A^2 + B^2 - 2AB\cos\frac{y}{x}\right)}}{\sin\frac{y}{x}}.$$

Also, show that for the line $y - b = mx$ the divisor becomes

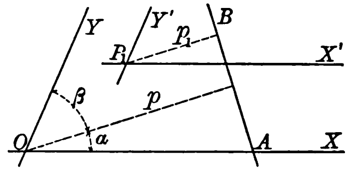
$$\frac{\sqrt{(1 + 2m\cos\omega + m^2)}}{\sin\omega}.$$

Proposition 12.†

145. ***Theorem.***—*The equation*

$$\pm (x_1 \cos \alpha + y_1 \cos \beta - p) = p_1$$

expresses the length of a perpendicular let fall from any point upon a given line; in which (x_1, y_1) *is the point,* p_1 *is the length of the perpendicular, and* $x \cos \alpha + y \cos \beta - p = 0$ *is the equation of the given line.*



For, if $x \cos \alpha + y \cos \beta - p = 0$ is (Art. 139) the equation of AB referred to O, and we move the origin to P_1 (Art. 23), the equation of AB then becomes

$$x_1 \cos \alpha + y_1 \cos \beta + x' \cos \alpha + y' \cos \beta - p = 0.$$

But $x' \cos \alpha + y' \cos \beta - p_1 = 0$ is the equation of AB referred to P_1. Substituting, we obtain $-(x_1 \cos \alpha + y_1 \cos \beta - p) = p_1$. When P_1 is on the side of AB opposite to the origin, evidently the perpendicular p_1 is negative, since it is let fall in a direction opposite to that of p,

$$\therefore \quad \pm (x_1 \cos \alpha + y_1 \cos \beta - p) = p_1.$$

146. ***Cor.***—By a reduction like that of Art. 142 it can be shown that

$$p_1 = \frac{Ax_1 + By_1 + C}{\pm \sqrt{(A^2 + B^2)}}.$$

147. ***Examples.***—(1.) Find the length of the perpendicular from the point $(7, 2)$ on the right line, $4x + 3y = 10$. *Ans.* $p_1 = 4.8$

(2.) Find the lengths of the perpendiculars from the point $(-4, 3)$ on the sides of the triangle whose vertices are $(1, 5)$, $(5, 1)$ and $(-1, -1)$.

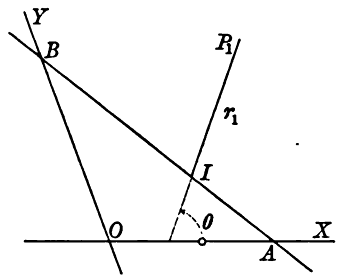
148. ***Exercise.***—Prove that the equation

$$r_1 = -\frac{Ax_1 + By_1 + C}{A \cos \theta + B \sin \theta}$$

expresses the length of line drawn in a given direction from a given point to meet a given line; in which r_1 is the length, $Ax + By + C = 0$ the given line, θ its inclination to the axis of x, and (x_1, y_1) the given point

We may also write

$$r_1 = -\frac{x_1 \cos{}^{l}_{x} + y_1 \cos{}^{l}_{y} - p}{\cos{}^{l}_{x} \cos\theta + \cos{}^{l}_{y} \sin\theta}$$



E. G. The length of a line drawn from the point $(-4, 10)$ to meet the line $x - 2y = 3$, and making an angle of $45°$ with the axis of x, will be found to be $r_1 = -38.05$.

Proposition 13.

149. ***Theorem.***—*The equation*

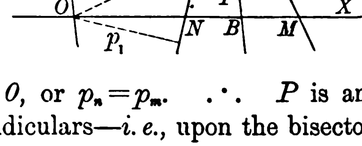
$$(x \cos{}^{p_1}_{x} + y \cos{}^{p_1}_{y} - p_1) \pm (x \cos{}^{p_2}_{x} + y \cos{}^{p_2}_{y} - p_2) = 0$$

is that of the line bisecting the angle between two given lines; in which

$$x \cos{}^{p_1}_{x} + y \cos{}^{p_1}_{y} - p_1 = 0, \quad \textit{and} \quad x \cos{}^{p_2}_{x} + y \cos{}^{p_2}_{y} - p_2 = 0$$

are the given lines.

For, by Art. 115 it is the equation of some right line as PP_2 through the intersection of two others, as Mm and Nn. Take any point (x, y) on the line (P, in the figure); then (Art. 145)



$$x \cos{}^{p_1}_{x} + y \cos{}^{p_1}_{y} - p_1 = +p_n$$

and

$$x \cos{}^{p_2}_{x} + y \cos{}^{p_2}_{y} - p_2 = -p_m.$$

But by hypothesis the sum of the first members of these equations $= 0$; $\therefore$ $p_n - p_m = 0$, or $p_n = p_m$. $\therefore$ P is any point on the line of equal perpendiculars—*i. e.*, upon the bisector.

$$\therefore \quad (x \cos{}^{p_1}_{x} + y \cos{}^{p_1}_{y} - p_1) + (x \cos{}^{p_2}_{x} + y \cos{}^{p_2}_{y} - p_2) = 0$$

is the equation of the *external* bisector—*i. e.*, the bisector of the angle not containing the origin, as BP_2. Similarly,

$$(x \cos{}^{p_1}_{x} + y \cos{}^{p_1}_{y} - p_1) - (x \cos{}^{p_2}_{x} + y \cos{}^{p_2}_{y} - p_2) = 0$$

is the equation of the *internal* bisector AP_2.

150. ***Cor. 1.***—The equation in rectangular co-ordinates of the bisector of the lines

$$A_1x + B_1y + C_1 = 0, \text{ and } A_2x + B_2y + C_2 = 0, \text{ is (Art. 142)}$$

$$\frac{A_1x + B_1y + C_1}{\sqrt{(A_1 + B_1)}} \pm \frac{A_2x + B_2y + C_2}{\sqrt{(A_2 + B_2)}} = 0$$

151. ***Cor. 2.***—The co-ordinates of the point of intersection of the lines $x \cos a_1 + y \sin a_1 - p_1 = 0$, and $x \cos a_2 + y \sin a_2 - p_2 = 0$, are (Art. 113), $x_i = \dfrac{p_1 \sin a_2 - p_2 \sin a_1}{\sin (a_1 - a_2)}$, and $y_i = \dfrac{p_1 \cos a_2 - p_2 \cos a_1}{\sin (a_1 - a_2)}$.

152. ***Cor. 3.***—The lines

$$x \cos a_1 + y \sin a_1 - p_1 = 0,$$
$$x \cos a_2 + y \sin a_2 - p_2 = 0,$$
$$x \cos a_3 + y \sin a_3 - p_3 = 0,$$

(Art. 117), intersect in a point, when

$$p_1 \sin (a_2 - a_3) + p_2 \sin (a_3 - a_1) + p_3 \sin (a_1 - a_2) = 0.$$

153. ***Cor. 4.***—By Art. 118 we have also,

$$\frac{[p_1 \sin (a_2 - a_3) + p_2 \sin (a_3 - a_1) + p_3 \sin (a_1 - a_2)]^2}{\sin (a_2 - a_3) \sin (a_3 - a_1) \sin (a_1 - a_2)} = \pm 2t.$$

154. ***Example.***—Form the equations—in rectangular co-ordinates—of the internal bisectors of the angles of a triangle, the co-ordinates of whose vertices are $(1, 2)$, $(-3, 5)$, and $(-1, -4)$; and show that they intersect in a common point.

Ans. $y = 0.312\,x + 1.692$, $y = -1.55\,x + 0.35$, $y = 19.39\,x + 15.39$

Co-ordinates of point of intersection.

$x = -0.72$, nearly, $y = 1.466$, nearly.

155. ***Exercise.***—Prove that in oblique co-ordinates

$$\frac{A_1x + B_1y + C_1}{\sqrt{\left(A_1^2 + B_1^2 - 2A_1B_1 \cos \frac{y}{x}\right)}} \pm \frac{A_2x + B_2y + C_2}{\sqrt{\left(A_2^2 + B_2^2 - 2A_2B_2 \cos \frac{y}{x}\right)}} = 0$$

is the equation of the bisectors of the lines

$$A_1x + B_1y + C_1 = 0, \text{ and } A_2x + B_2y + C_2 = 0.$$

POLAR CO-ORDINATES.

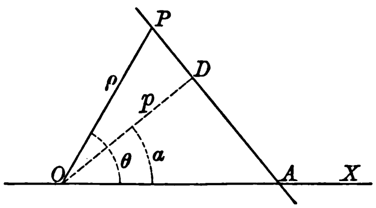
Proposition 14.

156. ***Theorem.****—The equation*

$$\rho \cos \left(\frac{\rho}{x} - \frac{p}{x}\right) = p$$

represents a right line; in which ρ ***is the radius vector of any point of the line, and*** p ***the length of the perpendicular from the pole upon the line.***

For, by trigonometry,



$\rho \cos \frac{\rho}{p} = p$. But $\frac{\rho}{p} = \frac{x}{p} + \frac{\rho}{x}$.

$\therefore \quad \rho \cos \left(\frac{\rho}{x} - \frac{p}{x}\right) = p.$

This may also be written

$$\rho \cos (\theta - \alpha) = p, \quad \text{or} \quad \rho \cos \alpha \cos \theta + \rho \sin \alpha \sin \theta = p.$$

157. ***Cor. 1.***—If $\alpha = 0$, then $\rho \cos 0 = p$ is the equation of a line perpendicular to the initial line.

158. ***Cor. 2.***—The equation $\theta = c$ represents a line through the origin making the angle c with the initial line.

159. ***Cor. 3.***—The angle between the lines $\rho \cos (\theta - \alpha_1) = p_1$ and $\rho \cos (\theta - \alpha_2) = p_2$ is the same as that between p_1 and p_2—*i. e.*,

$$\frac{l_2}{l_1} = \frac{p_2}{p_1} = \alpha_2 - \alpha_1.$$

160. ***Cor. 4.***—Two lines are perpendicular to each other if $\alpha_2 - \alpha_1 = 90°$, and parallel if $\alpha_2 - \alpha_1 = 0$.

161. ***Cor. 5.***—The equation of the *external* bisector between the lines (1) and (2) is (Arts. 83 and 149), when $\omega = 90°$,

$$\rho \left[\cos (\theta - \alpha_1) + \cos (\theta - \alpha_2)\right] = p_2 + p_1.$$

By trig. $\qquad \cos m + \cos n = 2 \cos \left(\frac{m+n}{2}\right) \cos \left(\frac{m-n}{2}\right).$

$$\therefore \quad \rho \cos\left(\frac{\theta - \alpha_2 + \alpha_1}{2}\right) \cos\left(\frac{\alpha_2 + \alpha_1}{2}\right) = \frac{p_2 + p_1}{2}.$$

Similarly for the *internal* bisector,

$$\rho \cos\left(\frac{\theta - \alpha_2 + \alpha_1 + \pi}{2}\right) \sin\left(\frac{\alpha_2 - \alpha_1}{2}\right) = \frac{p_2 - p_1}{2}.$$

162. ***Schol. 1.***—The equation $\rho \cos(\theta - \alpha) = \rho_1 \cos(\theta_1 - \alpha)$ is the polar equation of a line through the point (ρ_1, θ_1), for, (Art. 156), each side of the equation $= p$.

163. ***Schol. 2.***—The equation (Arts. 83 and 91)

$$\frac{\rho \cos\theta - \rho_1 \cos\theta_1}{\rho_1 \cos\theta_1 - \rho_2 \cos\theta_2} = \frac{\rho \sin\theta - \rho_1 \sin\theta_1}{\rho_1 \sin\theta_1 - \rho_2 \sin\theta_2},$$

or, $\quad \rho\,[\rho_1 \sin(\theta - \theta_1) - \rho_2 \sin(\theta - \theta_2)] = -\rho_1 \rho_2 \sin(\theta_1 - \theta_2)$

is the polar equation of a line through two given points (ρ_1, θ_1) and (ρ_2, θ_2).

Proposition 15.

164. ***Theorem.***—*The equation*

$$A\,\rho \cos\theta + B\,\rho \sin\theta + C = 0$$

represents a right line; in which ρ and θ are the polar co-ordinates of any point in the line.

For, transforming to rectangulars, since, by Art. 83, $x = \rho \cos\theta$, and $y = \rho \sin\theta$, the equation becomes $Ax + By + C = 0$, which is the equation of a right line by Art. 109.

165. ***Schol. 1.***—$A\rho \cos\theta + B\rho \sin\theta + C = 0$ can be reduced to the form $\rho \cos(\theta - \alpha) = p$; for by Art. 142,

$$\frac{-A}{\sqrt{(A^2 + B^2)}}\,\rho \cos\theta + \frac{-B}{\sqrt{(A^2 + B^2)}}\,\rho \sin\theta = \frac{C}{\sqrt{(A^2 + B^2)}}$$

is of the form $\rho(\cos\alpha \cos\theta + \sin\alpha \sin\theta) = p$.

$$\therefore \quad \rho \cos(\theta - \alpha) = p, \text{ when } \tan\alpha = \frac{B}{A}, \text{ and } p = \frac{C}{\sqrt{(A^2 + B^2)}}.$$

166. ***Schol. 2.***—The condition that three right lines given by their polar equations, pass through one point is found in Art. 152, and the area enclosed by three right lines in Arts. 54 and 153.

167. ***Exercises.***—Let the vertices of a triangle be (x_1, y_1), (x_2, y_2), and (x_3, y_3).

(1.) Find the equations of its sides (Art. 90).

(2.) Show (Arts. 37 and 128) that the equations of the perpendiculars which bisect its sides are

$$(x_1 - x_2)\,x + (y_1 - y_2)\,y = \tfrac{1}{2}(x_1^2 - x_2^2) + \tfrac{1}{2}(y_1^2 - y_2^2)$$
$$(x_2 - x_3)\,x + (y_2 - y_3)\,y = \tfrac{1}{2}(x_2^2 - x_3^2) + \tfrac{1}{2}(y_2^2 - y_3^2)$$
$$(x_3 - x_1)\,x + (y_3 - y_1)\,y = \tfrac{1}{2}(x_3^2 - x_1^2) + \tfrac{1}{2}(y_3^2 - y_1^2).$$

(3.) Show (Art. 116) that these three perpendiculars pass through one point, and find the co-ordinates of the point of intersection.

(4.) Show that the equations of the lines through the vertices and perpendicular to the sides opposite them are

$$(x_1 - x_2)\,x + (y_1 - y_2)\,y = (x_1 - x_2)\,x_3 + (y_1 - y_2)\,y_3$$
$$(x_2 - x_3)\,x + (y_2 - y_3)\,y = (x_2 - x_3)\,x_1 + (y_2 - y_3)\,y_1$$
$$(x_3 - x_1)\,x + (y_3 - y_1)\,y = (x_3 - x_1)\,x_2 + (y_3 - y_1)\,y_2.$$

(5.) Show that these three perpendiculars also pass through one point, and find its co-ordinates.

(6.) Show (Art. 90) that the equations of the lines through the vertices and bisecting the sides opposite them are

$$(y_1 + y_2 - 2y_3)\,x - (x_1 + x_2 - 2x_3)\,y = (y_1 + y_2)\,x_3 - (x_1 + x_2)\,y_3$$
$$(y_2 + y_3 - 2y_1)\,x - (x_2 + x_3 - 2x_1)\,y = (y_2 + y_3)\,x_1 - (x_2 + x_3)\,y_1$$
$$(y_3 + y_1 - 2y_2)\,x - (x_3 + x_1 - 2x_2)\,y = (y_3 + y_1)\,x_2 - (x_3 + x_1)\,y_2$$

(7.) Show that these three bisecting lines pass through a single point, and that its co-ordinates are

$$x_i = \tfrac{1}{3}(x_1 + x_2 + x_3), \quad \text{and} \quad y_i = \tfrac{1}{3}(y_1 + y_2 + y_3).$$

(8.) Show that the equations of the three bisectors of the angles of a triangle are

$$(x \cos a_1 + y \sin a_1 - p_1) - (x \cos a_2 + y \sin a_2 - p_2) = 0$$

$$(x \cos a_2 + y \sin a_2 - p_2) - (x \cos a_3 + y \sin x_3 - p_3) = 0$$

$$(x \cos a_3 + y \sin a_3 - p_3) - (x \cos a_1 + y \sin a_1 - p_1) = 0.$$

(9.) Show that these bisectors intersect in a point, and find the co-ordinates of the point.

(10.) Show how many of the points mentioned in (3), (5), (7) and (9) are upon the same straight line (Art. 91).

(11.) Show (Arts. 146 and 148) that when a line cuts the sides of a triangle ABC (produced if necessary) in the points L, M and N, then

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1.$$

N. B.—Such a line as LN is a *transversal.*

(12.) Show (Art. 31) that when lines be drawn from any point through the vertices of the triangle ABC, and meeting the opposite sides, BC, CA, and AB respectively in D, E and F, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = +1.$$

N. B.—The three lines meeting at a point are *convergents.*

(13.) Show (Art. 151) that the polar co-ordinates of the intersection of two right lines are

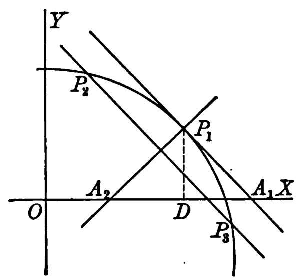
$$\rho_i = \frac{\sqrt{p_1^2 + p_2^2 - 2p_1p_2 \cos(a_1 - a_2)}}{\sin(a_1 - a_2)},$$

$$\tan \theta_i = \frac{p_1 \cos a_2 - p_2 \cos a_1}{p_1 \sin a_2 - p_2 \sin a_1}.$$

CHAPTER IV.

THE CIRCLE.

168. Tangent Line.—If a secant line be passed through two points, P_2 and P_3, of a curve, and the points be conceived to move continuously along the curve until they meet at P_1 (the secant continuing to pass through the points in all their positions), the secant in its limiting position is called a *tangent* to the curve at P_1, and is said to touch it in two consecutive or coincident points at P_1.



The equation of the secant or chord P_2P_3 is (Art. 90), $\dfrac{y - y_2}{x - x_2} = \dfrac{y_3 - y_2}{x_3 - x_2}$. When the value of the *second member* of this equation is determined from the equation of any particular curve, and substituted, and the points are then made *consecutive*, the equation becomes that of the *tangent to the curve.** The intercept on x is $OA_1 = a_1$. The subtangent is DA_1, and its length is $a_1 - x_1$.

The length of the tangent is $P_1A_1 = \sqrt{y_1^2 + (a_1 - x_1)^2}$.

169. Normal Line.—A normal line to a curve is so situated that it is *perpendicular* to the tangent at the point of contact.

The equation of the normal can be obtained from that of the tangent by finding the equation of a line through the point of contact, and perpendicular to the tangent (Art. 128).** The intercept on x is

* In Differential Calculus the general equation of the tangent line for all curves is $\dfrac{y-y_1}{x-x_1} = \dfrac{dy_1}{dx_1}$. ** The general equation of the normal line is $\dfrac{y-y_1}{x-x_1} = -\dfrac{dx_1}{dy_1}$.

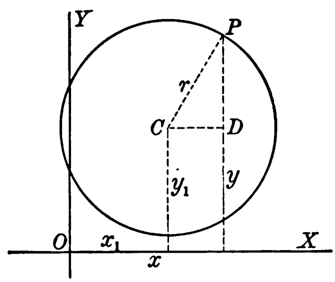
$OA_2 = a_2$; the subnormal is A_2D, and its length is $x_1 - a_2$. The length of the normal is $P_1A_2 = \sqrt{y_1^2 + (x_1 - a_2)^2}$.

Proposition 1.

170. ***Theorem.—The equation***

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

represents a circle; in which** x **and** y **are the rectangular co-ordinates of any point of the circumference,** x_1 **and** y_1 **those of the centre, and** r **is the length of the radius.



For, by Art. 26 in the equation

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2,$$

r represents the distance between (x_1, y_1) and (x_2, y_2). Let (x_2, y_2) be every point at the distance r from the fixed point (x_1, y_1); then if x and y represent the general values of x_2 and y_2, the equation becomes $(x - x_1)^2 + (y - y_1)^2 = r^2$. But the point (x, y) is everywhere at the distance r from (x_1, y_1), and hence must be always in the circumference of a circle whose radius is r. The same equation can be proved directly from the figure.

171. ***Schol. 1.***—If the origin be at the centre then $x_1 = 0$, and $y_1 = 0$,

$$\therefore\ x^2 + y^2 = r^2.$$

If the origin be on the circumference, we have $x_1^2 + y_1^2 = r^2$;

substitute $\qquad \therefore\ x^2 + y^2 - 2x_1x - 2y_1y = 0.$

If in addition, the axis of x pass through the centre, then $y_1 = 0$, and $\quad \therefore\ x^2 + y^2 - 2x_1x = 0$; or since now $x_1 = r$, by transposition,

$$y^2 = 2rx - x^2.$$

If the axis of y pass through the centre, $x_1 = 0$.

$\therefore\ x^2 + y^2 - 2y_1y = 0$; or since now $y_1 = r$, $\quad \therefore\ x^2 = 2ry - y^2$.

These are all of the form $\quad x^2 + y^2 + Ax + By + C = 0.$

172. ***Schol. 2.***—Conversely, the equation $x^2+y^2+Ax+By+C=0$, referred to *rectangular axes*, always represents a *circle;* for it may be written

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)=\frac{A^2+B^2-4C}{4},$$

which is of the form $(x-x_1)^2+(y-y_1)^2=r^2$,

in which $x_1=-\frac{A}{2}$, and $y_1=-\frac{B}{2}$.

If $A^2+B^2-4C>0$, the circle has $r=\frac{1}{2}\sqrt{A^2+B^2-4C}$, and the point $\left(-\frac{A}{2}, -\frac{B}{2}\right)$ for its centre.

If $A^2+B^2-4C=0$, the circle is the *point* (x_1, y_1).

If $A^2+B^2-4C<0$, the circle is *imaginary.*

173. ***Examples.***—Construct the circles represented by the following equations.

(1.) $x^2+y^2-4x=5.$

(2.) $x^2+y^2+6x-6y=-9.$

(3.) $x-y-x^2-y^2=0.$

(4.) $x^2+y^2+xy+3y=1.$

Find the radii and co-ordinates of the centres of the above circles.

174. ***Exercise.***—Show that the equation

$$x^2+y^2+2xy\cos\frac{y}{x}+Ax+By+C=0$$

represents a circle referred to oblique axes.

Proposition 2.†

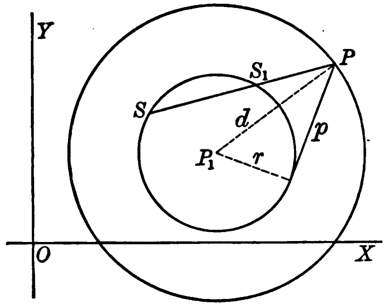
175. ***Theorem.***—*The equation*

$$(x-x_1)^2+(y-y_1)^2-r^2=\pm p^2$$

expresses the rectangle of the segments into which any line passing through a given point, and cutting a given circle,

is divided by that point; in which (x, y) is the given point, p^2 the area of the rectangle, and $(x-x_1)^2+(y-y_1)^2=r^2$ represents the given circle.

1st. Let P be *without* the given circle.

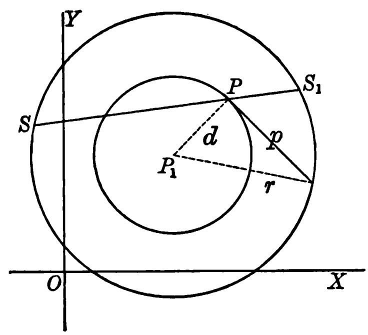


If $PP_1=d$, then $(x-x_1)^2+(y-y_1)^2=d^2$ . . . (*a.*) is the equation of the circle through P with centre P_1. If $d^2=r^2+p^2$, then p is the length of the tangent from P to the circle $(x-x_1)^2+(y-y_1)^2=r^2$.

$\therefore$ Substituting, eq. (*a.*) becomes $(x-x_1)^2+(y-y_1)^2=r^2+p^2$. But, by elementary geometry, $p^2=PS_1 \cdot PS_2$.

S_1 and S_2 are in the *same direction* from P, and the rectangle $PS_1 \cdot PS_2$ is therefore *positive.*

2d. Let P be *within* the given circle.



If $PP_1=d$, then $(x-x_1)^2+(y-y_1)^2=d^2$ . . . (*b.*) is the equation of the circle through P with the centre P_1. If $d^2=r^2-p^2$, then p is the length of the tangent from P to its intersection with the circle $(x-x_1)^2+(y-y_1)^2=r^2$.

$\therefore$ Substituting, eq. (*b.*) becomes $(x-x_1)^2+(y-y_1)^2=r^2-p^2$. But, $-p^2=PS_1 \cdot PS_2$, by elementary geometry.

S_1 and S_2 are in opposite directions from P; the rectangle of PS_1 and PS_2 is therefore negative.

176. ***Example.***—Find the rectangle of the segments of the secant, drawn from the point $(-2, 5)$, cutting the circle $4-2x^2-2y^2=x$.

Ans. 26.

Proposition 3 †.

177. ***Theorem.****—The equation*

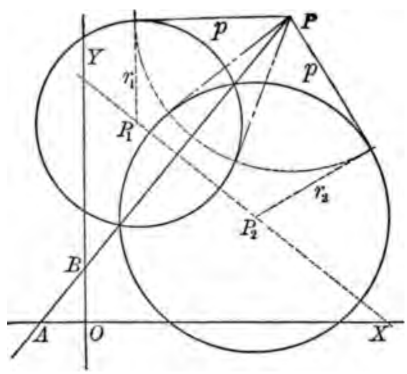
$$(x-x_1)^2+(y-y_1)^2-r_1^2=(x-x_2)^2+(y-y_2)^2-r_2^2$$

represents a right line, called the axis radical of the two circles whose radii are r_1 and r_2, and whose centres are at (x_1, y_1) and (x_2, y_2) respectively.

For, expand and cancel, and we have,

$$2(x_2-x_1)x+2(y_2-y_1)y \\ -(x_2^2-x_1^2)-(y_2^2-y_1^2) \\ +r_2^2-r_1^2=0,$$

which (Art. 109) is the equation of a right line. If some point P—*i. e.*, (x, y)—be taken on this line without both circles, then, by Art. 175,



$$(x-x_1)^2+(y-y_1)^2-r_1^2=(x-x_2)^2+(y-y_2)^2-r_2^2=p^2,$$

which expresses the property of the axis radical, that the tangent drawn from any point of the line to one circle, is equal to the tangent drawn from the same point to the other circle. The line is real whether the circles intersect or not; and if it passes within one circle, it passes within the other at the same time.

178. ***Schol. 1.***—The equation to the line of centres is (Art. 90)

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1).$$

The equation to the axis radical is

$$y=-\frac{x_2-x_1}{y_2-y_1}\,.\,x+C,$$

hence these two lines are perpendicular to each other.

179. ***Schol. 2.***—The equations of the axes radical of three circles

taken two and two whose centres are the points (x_1, y_1), (x_2, y_2), (x_3, y_3), are (Art. 177)

$$(x_1 - x_2)\,x + (y_1 - y_2)\,y = \tfrac{1}{2}\left[(x_1^2 - x_2^2) + (y_1^2 - y_2^2) - (r_1^2 - r_2^2)\right]$$

$$(x_2 - x_3)\,x + (y_2 - y_3)\,y = \tfrac{1}{2}\left[(x_2^2 - x_3^2) + (y_2^2 - y_3^2) - (r_2^2 - r_3^2)\right]$$

$$(x_3 - x_1)\,x + (y_3 - y_1)\,y = \tfrac{1}{2}\left[(x_3^2 - x_1^2) + (y_3^2 - y_1^2) - (r_3^2 - r_1^2)\right]$$

which intersect in a single point called the (Art. 116) *centre radical.* Compare these equations with the equations of the parallels to them. Art. 167 (2). This may also be more easily demonstrated as follows: if by the equation $S_1 = 0$ be understood $(x - x_1)^2 + (y - y_1)^2 - r_1^2 = 0$, then will $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, represent three circles, and the equations $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$, will be those of their axes radical, which meet in a single point (Art. 116).

180. ***Example.***—Find the axes radical of three circles, two by two, whose radii are respectively 2, 3 and 4, and whose centres are at the points $(1, 2)$, $(5, -2)$ and $(-1, -3)$; and show that the three intersect in a common point. *Ans.* $8x - 8y = 19.$

$$4x + 10y = 7.$$

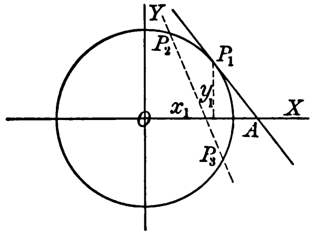
$$6x + y = 13.$$

Proposition 4.

181. ***Theorem.—The equation***

$$xx_1 + yy_1 = r^2$$

represents a line tangent to the circle $x^2 + y^2 = r^2$; ***in which*** (x_1, y_1) ***is the point of tangency.***



For, if two points P_2 and P_3 be taken upon the circumference of the circle $x^2 + y^2 = r^2$, the equations $x_2^2 + y_2^2 = r^2$ and $x_3^2 + y_3^2 = r^2$ are the equations of condition that P_2 and P_3 be so situated.

The equation of the line P_2P_3 is (Art. 90) $\dfrac{y - y_2}{x - x_2} = \dfrac{y_3 - y_2}{x_3 - x_2}$. (*a.*)

But $$x_2^2 + y_2^2 = x_3^2 + y_3^2 = r^2.$$

$$\therefore \quad x_3^2 - x_2^2 + y_3^2 - y_2^2 = 0.$$

$$\therefore \quad (x_3 - x_2)(x_3 + x_2) + (y_3 - y_2)(y_3 + y_2) = 0.$$

$$\therefore \quad \frac{y_3 - y_2}{x_3 - x_2} = -\frac{x_3 + x_2}{y_3 + y_2}.$$

Substituting this value in eq. (*a.*) we have

$$\frac{y - y_2}{x - x_2} = -\frac{x_3 + x_2}{y_3 + y_2} \quad \ldots \; (b.),$$

as the equation of the line secant to the circle $x^2 + y^2 = r^2$, through P_2 and P_3. Let P_2 and P_3 be conceived to approach each other along the curve until they are consecutive points at P_1 (Art. 168). The secant line passing through these consecutive points will be the tangent, and we shall have, $x_3 = x_2 = x_1$, and $y_3 = y_2 = y_1$.

$\therefore$ eq. (*b.*) becomes $\dfrac{y - y_1}{x - x_1} = -\dfrac{x_1}{y_1}$, or $yy_1 - y_1^2 = -xx_1 + x_1^2$.

$$\therefore \quad xx_1 + yy_1 = x_1^2 + y_1^2 = r^2.$$

182. ***Cor. 1.***—If the origin be changed (Art. 23), the equation becomes $(x - x_0)(x_1 - x_0) + (y - y_0)(y_1 - y_0) = r^2$, which is tangent to the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, at the point (x_1, y_1).

183. ***Cor. 2.***—The intercepts of the tangent upon the axes are

$$\text{when } y = 0, \quad x = \frac{r^2 + x_0(x_1 - x_0) + y_0(y_1 - y_0)}{x_1 - x_0} = a;$$

$$\text{when } x = 0, \quad y = \frac{r^2 + x_0(x_1 - x_0) + y_0(y_1 - y_0)}{y_1 - y_0} = b$$

184. ***Cor. 3.***—The *subtangent*

$$= a - x_1 = \frac{r^2 + y_0(y_1 - y_0) - (x_1 - x_0)^2}{x_1 - x_0}.$$

When the origin is at the centre, the subtangent $= \dfrac{r^2}{x_1} - x_1 = \dfrac{r^2 - x_1^2}{x_1}$.

185. ***Schol.***—The equation of the tangent at (x_1, y_1) (see fig. of Art. 187) is

$$x_1x + y_1y = r^2,$$

and that of the tangent at the other extremity of the diameter—*i. e.*, at $(-x_1, -y_1)$—is

$$-x_1x - y_1y = r^2,$$

$$\therefore \quad x_1x + y_1y = \pm r^2, \text{ or } y + \frac{x_1}{y_1}x = \pm r\frac{r}{y_1}$$

represents the tangent through (x_1, y_1), and also the only tangent parallel to it.

$$\text{Let} \quad XOP_1 = \theta_1, \text{ then } \frac{x_1}{y_1} = \tan\theta_1 = m,$$

and by trig. $\dfrac{r}{y_1} = \sec\theta_1 = \sqrt{\tan^2\theta_1 + 1} = \sqrt{m^2 + 1}.$

$$\therefore \quad y + mx = \pm r\sqrt{m^2 + 1},$$

or when the origin is not at the centre (Art. 23),

$$y - y_0 + m(x - x_0) = \pm r\sqrt{m^2 + 1}$$

represents the parallel tangents to the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, and is often called their magic equation.

186. ***Examples.***—Form the equations to the following tangent lines.

(1.) Circle $x^2 + y^2 = 9$, at the point $(2.4, 1.8)$.

Ans. $3y + 4x = 15$.

(2.) Circle $x^2 + y^2 - 10x - 12y = -57$, at the point $(4.5, 4.06)$.

Ans. $x + 3.88y = 20.28$.

(3.) Form the equations to the tangent lines parallel to those in examples (1) and (2).

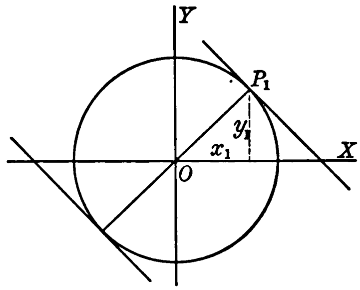
(4.) Find the intercepts of the tangents in examples (1), (2) and (3), and their subtangents.

Proposition 5.

187. ***Theorem.***—*The equation*

$$x_1y = xy_1$$

represents a right line normal to the circle $x^2 + y^2 = r^2$, *at the point* (x_1, y_1).



For, resuming the equation of the line tangent to the circle $x^2 + y^2 = r^2$ at P_1, viz.:

(Art. 181) $$\frac{y-y_1}{x-x_1} = -\frac{x_1}{y_1},$$

by Art. 123, $$\frac{y-y_1}{x-x_1} = +\frac{y_1}{x_1}$$

is perpendicular to the tangent at P_1; it is therefore the normal.

$$\therefore \quad x_1y - x_1y_1 = xy_1 - x_1y_1, \qquad \therefore \quad x_1y = xy_1.$$

Since this line passes through the origin, all normals to the circle pass through its centre.

188. ***Cor. 1.***—If the origin be changed, the equation becomes

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0},$$

which is evidently the equation of a line through P_1 and the centre P_0, and is normal to the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$.

189. ***Cor. 2.***—The intercepts of the normal are,

when $y = 0$, $\quad x = \frac{x_0y_1 - x_1y_0}{y_1 - y_0} = a$

when $x = 0$, $\quad y = \frac{x_1y_0 - x_0y_1}{x_1 - x_0} = b.$

The *subnormal* $= a - x_1 = \frac{-y_1(x_1 - x_0)}{y_1 - y_0}$

190. ***Example.***—Form the equation of the line normal to the circle $x^2 + y^2 - 4x + y + 2 = 0$, at the point $(4, 5)$. *Ans.* $4y + 24 = 11x$.

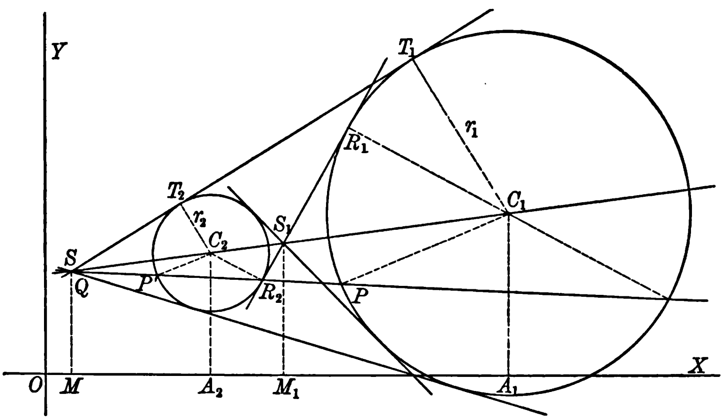
Proposition 6.†

191. ***Theorem.***—*The equations*

$$x_s = \frac{r_1x_2 \mp r_2x_1}{r_1 \mp r_2} \text{ and } y_s = \frac{r_1y_2 \mp r_2y_1}{r_1 \mp r_2}$$

express the positions of the intersections of the common tangents of two circles—i. e., their ***centres of similitude;*** *in which r_1 and r_2 are the radii, and (x_1, y_1), (x_2, y_2) are the centres of the circles.*

By geometry S and S_1 fall on C_2C_1.



By similarity of triangles

$r_1 : r_2 : : ST_1 : ST_2$

$: : SC_1 : SC_2$

$: : MA_1 : MA_2$. $\therefore\ r_1 : r_2 : : x_s - x_1 : x_s - x_2$

$\therefore$ solving $$x_s = \frac{r_1x_2 - r_2x_1}{r_1 - r_2},$$

and similarly, $$y_s = \frac{r_1y_2 - r_2y_1}{r_1 - r_2}.$$

Again, $$r_1 : r_2 :: R_1S_1 : S_1R_2$$
$$:: C_1S_1 : S_1C_2$$
$$:: A_1M_1 : M_1A_2$$
$$\therefore \quad r_1 : r_2 :: x_s - x_1 : x_2 - x_s$$

and solving, $$x_s = \frac{r_1x_2 + r_2x_1}{r_1 + r_2},$$

and similarly, $$y_s = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

Compare the equations of Art. 37.

192. ***Cor.***—Draw any parallel radii, as C_1P, C_2P', then PP' cuts C_1C_2 in some point Q—*i. e.*, (x_q, y_q).

$$\therefore \quad r_1 : r_2 :: QP : QP'$$
$$:: QC_1 : QC_2$$
$$\therefore \quad r_1 : r_2 :: x_q - x_1 : x_q - x_2$$
$$\therefore \quad x_q = \frac{r_1x_2 - r_2x_1}{r_1 - r_2}$$
$$\therefore \quad x_s = x_q, \text{ and similarly } y_s = y_q.$$

The same may be proved respecting S_1. Hence all lines drawn through the extremities of parallel radii pass through S or S_1, and are cut by the circles, so that $r_1 : r_2 :: SP : SP'$. For this reason S is called the *external* and S_1 the *internal centre of similitude.*

193. ***Schol. 1.***—It is possible to obtain the equations of the four common tangents to the two circles $(x - x_1)^2 + (y - y_1)^2 = r_1^2$, and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$ from Art. 185, by finding the four values of m to be obtained from the conditions $m = m_1 = m_2$ and

$$y_2 - y_1 + m_2x_2 - m_1x_1 \pm r_2\sqrt{m_2^2 + 1} \mp r_1\sqrt{m_1^2 + 1} = 0,$$

and then in the general equation of the tangent to each circle, each of which is of the form, $y - y_0 + m(x - x_0) = \pm r\sqrt{m^2 + 1}$, substituting these values: equations will thus be obtained, four of which will be identical.

E. G., The four common tangents to the circles

$$(y-2)^2+(x-5)^2=9$$
$$(y-1)^2+x^2=4$$

are obtained by the general equations (Art. 185),

$$y-2+m\,(x-5)=\pm 3\ \sqrt{m^2+1}\ldots(a.)$$
$$y-1+mx+\pm 2\ \sqrt{m^2+1}\ldots\ldots\ldots(b.)$$

becoming identical.

Eliminate x and y from (*a.*) and (*b.*), and we find the four values

$$m=2.4,\quad m=\infty,\quad m=-\frac{5}{12},\quad m=0,$$

and (*a.*) becomes, by substituting these four values of m,

$$y+2.4x=6.2,\text{ or }=21.8$$
$$y=2,\text{ or }=8$$
$$y-\frac{5}{12}x=\frac{19}{12},\text{ or }=-\frac{10}{3}$$
$$y=-1,\text{ or }=5$$

and (*b.*) becomes in a similar manner,

$$y+2.4x=6.2,\text{ or }=-4.2$$
$$y=2,\text{ or }=-2$$
$$y-\frac{5}{12}x=\frac{19}{12},\text{ or }=-\frac{7}{6}$$
$$y=-1,\text{ or }=3.$$

Hence the four common tangents are,

$$y+2.4x=6.2$$
$$y=2$$
$$12y=5x+19$$
$$y=-1.$$

194. ***Schol. 2.***—The external centres of similitude of three circles, taken two and two, lie on one line called the *axis of similitude.* For the centres of similitude

$$\left(\frac{r_1x_2-r_2x_1}{r_1-r_2},\ \frac{r_1y_2-r_2y_1}{r_1-r_2}\right)$$

$$\left(\frac{r_2x_3-r_3x_2}{r_2-r_3},\ \frac{r_2y_3-r_3y_2}{r_2-r_3}\right)$$

$$\left(\frac{r_3x_1-r_1x_3}{r_3-r_1},\ \frac{r_3y_1-r_1y_3}{r_3-r_1}\right)$$

will be found by substitution to fulfill the equation of condition of Art. 91.

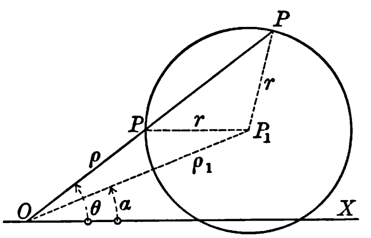
There are also, as may be proved in like manner, three internal axes of similitude, on each of which falls one external and two internal centres of similitude. Construct the figure by finding the intersections of the common tangents of three non-intersecting circles and drawing the axes of similitude.

Proposition 7.

195. ***Theorem.***—*The equation*

$$\rho^2-2\rho_1\rho\cos{}^{\rho}_{\rho_1}=r^2-\rho_1^2$$

represents a circle with radius r; in which ρ is the radius vector of any point of the circle, and ρ_1 that of the centre.



By trigonometry,

$$\rho^2+\rho_1^2-2\rho\rho_1\cos{}^{\rho}_{\rho_1}=r^2.\ .\ .\ .\ (a.)$$

Or if the axis of x is the initial line, we may write since

$$^{\rho}_{\rho_1}={}^{x}_{\rho_1}+{}^{\rho}_{x}={}^{\rho}_{x}-{}^{\rho_1}_{x}=\theta-a,$$

$$\therefore\ \ \rho^2-2\rho\rho_1\cos\left({}^{\rho}_{x}-{}^{\rho_1}_{x}\right)=r^2-\rho_1^2,$$

or, $\rho^2-2\rho\rho_1\cos(\theta-a)=r^2-\rho_1^2.\ .\ .\ (b.)$

This equation may also be obtained by transformation.

196. ***Schol.***—The general form of the polar equation of the circle is

$$\rho^2 + A\rho \cos \theta + B\rho \sin \theta + C = 0,$$

for, by trig. we may write eq. (***b.***)

$$\rho^2 - 2\rho\rho_1 (\cos \theta \cos \alpha + \sin \theta \sin \alpha) - (r^2 - \rho_1^2) = 0,$$

or, $\rho^2 - (2\rho_1 \cos \alpha)\, \rho \cos \theta - (2\rho_1 \sin \alpha)\, \rho \sin \theta - (r^2 - \rho_1^2) = 0.$

Now place the constants,

$$-2\rho_1 \cos \alpha = A, \quad -2\rho_1 \sin \alpha = B, \text{ and } \quad -(r^2 - \rho_1^2) = C;$$

$$\therefore \qquad \rho^2 + A\rho \cos \theta + B\rho \sin \theta + C = 0.$$

197. ***Cor.***—If the pole be upon the *circumference*, $\rho_1 = r$,

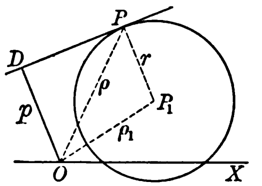
$\therefore \quad \rho^2 = 2\rho r \cos \frac{\rho}{r}$, which has the two roots, $\rho = 0$, and $\rho = 2r \cos \frac{\rho}{r}$.

The latter is then the polar equation of the circle.

If, however, the pole is at the centre, $\rho_1 = 0$, and $\rho^2 = r^2$,

$\therefore \quad \rho = r$ is then the polar equation of the circle.

198. ***Exercises.***—(1.) Prove that the perpendicular from the origin upon the tangent $p = \dfrac{r^2 + \rho^2 - \rho_1^2}{2r}.$



(2.) Show (Art. 172) that the equation

$$x^2 + y^2 - (x_1^2 + y_1^2) + A(x - x_1) + B(y - y_1) = 0$$

represents some circle through (x_1, y_1).

Also find the equation when A or B is eliminated instead of C.

(3.) Show that the equation

$$\begin{aligned}&(x^2 + y^2)(x_1y_2 - x_2y_1) + C(y_1 - y_2)\,x \\ &+ (x_1^2 + y_1^2)(x_2y - xy_2) + C(x_2 - x_1)\,y \\ &+ (x_2^2 + y_2^2)(xy_1 - x_1y) + C(x_1y_2 - x_2y_1) = 0\end{aligned}$$

represents some circle through the two points (x_1, y_1) and (x_2, y_2).

Also when B and C are eliminated the same equation is

$$\begin{aligned}&(x^2 + y^2)(y_1 - y_2) + (x_1^2 + y_1^2)(y_2 - y) + (x_2^2 + y_2^2)(y - y_1) \\ &- A[(x - x_1)\,y_2 + (x_2 - x)\,y_1 + (x_1 - x_2)\,y] = 0.\end{aligned}$$

(4.) Show that the equation

$$\left.\begin{array}{l}(x^2+y^2)\,[(x_1-x_2)\,y_3+(x_3-x_1)\,y_2+(x_2-x_3)\,y_1]\\ -(x_1^2+y_1^2)\,[(x_2-x_3)\,y+(x-x_2)\,y_3+(x_3-x)\,y_2]\\ +(x_2^2+y_2^2)\,[(x_3-x)\,y_1+(x_1-x_3)\,y+(x-x_1)\,y_3]\\ -(x_3^2+y_3^2)\,[(x-x_1)\,y_2+(x_2-x)\,y_1+(x_1-x_2)\,y]\end{array}\right\}=0.$$

represents a circle through the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Also show that the equation of condition for four points to be upon the same circumference is expressed by substituting (x_4, y_4) for (x, y) in this equation.

(5.) Show (Art. 31) that the equation of condition of four points on one circumference (4) has this geometric significance:

$$\rho_1^2\overline{P_2P_3P_4}+\rho_3^2\overline{P_4P_1P_2}=\rho_2^2\overline{P_3P_4P_1}+\rho_4^2\overline{P_1P_2P_3}$$

when ρ_1, ρ_2, ρ_3, ρ_4 are the distances of the points P_1, P_2, P_3, P_4 on the same circumference, and $\overline{P_1\ P_2\ P_3}$ signifies the area of the triangle, $P_1\ P_2\ P_3$, etc.

(6.) Show that

$$x^2+y^2-(a_1+a_2)\,x+a_1a_2+By=0$$

represents the circle whose intercepts on the axis of x are a_1 and a_2.

(7.) Show that

$$x^2+y^2+Ax-(b_1+b_2)\,y+b_1b_2=0$$

represents the circle whose intercepts on the axis of y are b_1 and b_2.

(8.) Show that

$$x^2+y^2-(a_1+a_2)\,x-(b_1+b_2)\,y+\tfrac{1}{2}(a_1a_2+b_1b_2)=0$$

represents the circle whose intercepts on the axes are a_1, a_2, b_1 and b_2.

(9.) Find the conditions that a circle touch each and both of the axes.

CHAPTER V.

EQUATIONS AND LOCI.

199. A plane locus is the line, straight or curved, described by a point moving on a plane according to some law.

This law consists in the existence of a *fixed relation* between the x and y co-ordinates of the point, in all its successive positions. *An equation of a locus* is used to express this law.

200. The connection between a **locus** and its **equation** is the *fundamental idea* of co-ordinate geometry.

E. G. It was shown (Art. 109) that the locus of a point moving according to a law expressed by any equation of the first degree in x and y is some right line. Thus, in the equation

$Ax+By+C=0$, let $A=1$, $B=1$ and $C=2$, $\therefore\ x+y+2=0$,

if any value whatever be assigned to x, a corresponding value of y will be obtained from the equation; and the points obtained by assigning successive values to x will be found to be all situated upon a fixed right line.

201. Variables.—The co-ordinates x and y are called *variables*, or sometimes *current* or *running* co-ordinates.

202. Constants.—The quantities A, B and C are called *constants.* When *given,* they cause the locus to be *definite* and *fixed.* If *varied,* they move the locus to some *new position* or effect some *change of form.* When *unknown* or *undetermined,* they are sometimes called *variable parameters* or *arbitrary constants.* Constants are *co-efficients* or *exponents* of variables.

203. The **equation** of a locus is composed, then, of two kinds of quantities, variables and constants.

Variables are the co-ordinates of a *moving point*, as (x, y), (ρ, θ), etc., for general values; (x_1, y_1), (ρ_1, θ_1), etc., for restricted values (Art. 19).

Constants are the co-ordinates of a *locus*.

N. B.—We have found it convenient to use A, B, C, m, p, a, b, c, etc., for general values of constants, and A_1, B_1, C_1, m_1, p_1, etc., for restricted values. Also, x_1, y_1, ρ_1, θ_1, etc., are frequently considered to be *constants*.

204. *Constants* impose the *restrictions* to which variables are subject, and the *conditions* which a locus may fulfill.

205. A Condition is some *fixed relation* existing between two or more *loci*. An *equation of condition* is used to express this relation.

E. G., It was shown in Art. 123 that $m_1 = \dfrac{-1}{m_2}$ is the equation of condition, which is true whenever the lines $y = m_1 x + b_1$, and $y = m_2 x + b_2$ are perpendicular to each other.

206. Transformation of Co-ordinates is the reference of any fixed or moving point to a *new system* of co-ordinates.

207. Equations of Transformation express the relation between the *primitive* co-ordinates and the *new*, and contain new variables, primitive variables and constants.

208. Equations of numerical value are for the purposes of *computation*. When an equation of any kind becomes *determinate*, it is of this character, as, moreover, are the equations of ordinary algebra.

209. Cyclic symmetry exists in an expression when certain letters or suffixes are *exchanged* or *permuted* in a cycle (see Art. 34), to form its different terms or equations.

E. G., In the equation of Art. 34 the suffixes are permuted in a cycle of three. In the equations of Arts. 167 and 198 a kind of double cyclic symmetry is noticeable. A symmetric arrangement is a useful aid to the memory, and greatly facilitates algebraic work.

Proposition 1.

210. ***Theorem.***—***Every term of any equation whatever represents quantity of the same kind, be it volume, area, distance, mere number or other measurable thing.***

For the terms are connected by one of the signs $+$, $-$, $=$; and since it is impossible to increase or diminish anything except by a quantity of the same nature, or to affirm the equality of unlike things, the proposition must be true.

211. ***Schol.***—A *volume* is said to be of *three dimensions*, an *area* of *two*, a *distance* of *one*, and a *ratio*, or mere number, of *no* dimensions.

Space is at most of only three dimensions, but space of four, five or n dimensions may be used as a purely analytic conception.

Proposition 2.

212. ***Theorem.***—***Any single equation between x and y represents some plane locus.***

For it expresses some fixed relation of x and y, such that if x have a given value, y becomes known. By assuming successive consecutive values of x, and finding the corresponding consecutive values of y, we trace the motion of a point. A point so moving describes a locus (Art. 199).

213. ***Schol.***—The locus of an equation which has no terms except those which contain x and y passes through the origin. For let one of the successive values of x be $x=0$, then $y=0$; but these are the co-ordinates of the origin. *E. G.*, In the equation $y^2=4px$ (see figure in Art. 243), if $x=0$, then $y=0$. See also Arts. 109 and 171. It will hereafter be shown (Art. 458) that when in addition there are no terms of the first degree the locus passes twice through the origin;

and when there are no terms of the second or lower degrees, the locus passes three times through the origin, etc.

214. ***Examples.***—Show the truth of the above proposition numerically by constructing the loci represented by the following equations.

(1.) $y = \dfrac{x}{x-2}$ (3.) $y = x^{-2}$

(2.) $y = \dfrac{x-3}{x+2}$ (4.) $y = \dfrac{x-1}{4}$.

Proposition 3.

215. ***Theorem.—Two simultaneous equations between x and y, one of the m^{th} degree and the other of the n^{th} degree, represent, in general, mn definite points upon a plane, which are the mn intersections of the loci of the two equations, considered singly, as in Art. 212.***

For, the two equations represent, each singly, a locus. The values of x and y can be simultaneous—that is, can refer to the same points, only at the points of intersection of the two loci. If the x and y of both equations are the same, the two equations will be sufficient to find by elimination the definite values of x and y, which are the co-ordinates of the points of intersection. But by algebra the degree of the equation resulting from elimination between two equations, one of the m^{th} and the other of the n^{th} degree, is mn, and such an equation will have mn roots. There will therefore be mn definite values of x and y, and mn intersections of the two loci.

216. ***Schol. 1.***—To find the intercepts of any locus on the axis of x, let $y = 0$, which is the equation of the axis of x. If $x = 0$, we shall obtain the intercepts on y.

217. ***Schol. 2.***—Any number of these mn points may *coincide*, which will be indicated by a corresponding number of *equal roots.* Any number of the mn points may be *at the origin*, or *at infinity*, when the corresponding roots will be 0 or ∞. Any even number of the mn

points may be *imaginary*—that is, *impossible*—which will be indicated by corresponding pairs of *imaginary roots.*

218. ***Examples.***—Find the points of intersection of the loci represented by the following equations.

(1.) $$\frac{x^2}{4}+\frac{y^2}{16}=1 \quad \text{and} \quad x^2+y^2=9.$$

Ans. $$\left(+\sqrt{\frac{7}{3}}, +2\sqrt{\frac{5}{3}}\right), \quad \left(+\sqrt{\frac{7}{3}}, -2\sqrt{\frac{5}{3}}\right),$$
$$\left(-\sqrt{\frac{7}{3}}, +2\sqrt{\frac{5}{3}}\right) \text{ and } \left(-\sqrt{\frac{7}{3}}, -2\sqrt{\frac{5}{3}}\right).$$

(2.) $$y^2=x^3 \quad \text{and} \quad y^2=4x.$$

Ans. $$(0,\ 0), \quad (0,\ 0),$$
$$(2,\ +2\sqrt{2}), \quad (2,\ -2\sqrt{2}),$$
$$(-2,\ +2\sqrt{-2}), \quad (-2,\ -2\sqrt{-2}).$$

Proposition 4.

219. ***Theorem.***—*The sum or difference of two equations of loci, of the m^{th} and n^{th} degrees respectively, is, in general, an equation representing a locus passing through the mn points of intersection of the two loci.*

For, if $S_m=0$ be understood to be an equation of the m^{th} degree in x and y, and $S_n=0$ one of the n^{th} degree, and k any number, then $S_m \pm kS_n=0$ is the equation of some locus, since it contains x and y; and since, when $S_m=0$ and $S_n=0$, we have $S_m \pm kS_n=0$ also, the three equations are *simultaneous* at the points of intersection of $S_m=0$ and $S_n=0$.

220. ***Examples.***—Construct the loci represented by the following equations, and show that they pass through the points of intersection of the loci represented by their component equations.

(1.) $(x+y-1)+(x^2+y^2-4)=0.$

(2.) $(x+y-1)-(x^2+y^2-4)=0.$

(3.) $3(x+y-1)-(x^2+y^2-4)=0.$

(4.) $(x^2+y^2+2x-3)-2(x^2+y^2-8x+7)=0.$

Arts. 115 and 177 also furnish examples under this proposition.

Proposition 5.†

221. ***Theorem.***—*Any equation of a locus, some of whose constants are general (that is, their values are undetermined), can, in general, be made to fulfill as many conditions as there are independent, undetermined constants in the equation.*

For, a *condition* may be expressed by an *equation* containing some one or more undetermined constants, and *two* conditions by *two* or more such equations; *one* equation suffices to determine *one* constant, or *two* equations *two* constants, etc. There can then be as many conditions—that is, equations—as there are undetermined constants.

222. ***Schol. 1.***—Any locus whose equation contains undetermined constants can, in general, be made to pass through as many given points as there are *independent, undetermined* constants in its equation. For, if the co-ordinates of one point, as (x_1, y_1), satisfy the equation, then, when x_1 and y_1 are substituted for x and y in the equation of the locus, the equation becomes the equation of condition that x_1 and y_1 shall be on the locus. Between this equation of condition and the equation of the locus, one constant can be eliminated. The same process for a second point would eliminate a second constant, and so on.

223. ***E. G.*** Take the equation of a right line $\frac{x}{a}+\frac{y}{b}=1, \quad . \; . \; . \quad (a.)$

and subject it to the condition of passing through the points (x_1, y_1) and (x_2, y_2). If (x_1, y_1) is on the line we must have

$$\frac{x_1}{a}+\frac{y_1}{b}=1, \quad . \; . \; . \quad (b.)$$

and eliminating between equations (*a.*) and (*b.*), we obtain

$$\frac{x}{a}+\frac{(a-x_1)\,y}{ay_1}=1. \quad . \quad . \quad (c.)$$

If (x_2, y_2) is on the line, we must have

$$\frac{x_2}{a}+\frac{y_2}{b}=1, \quad . \quad . \quad (d.)$$

and eliminating between (*a.*) and (*d.*), we find similarly

$$\frac{x}{a}+\frac{(a-x_2)\,y}{ay_2}=1. \quad . \quad . \quad (e.)$$

Now eliminate a between equations (*c.*) and (*e.*), and we have

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

as in Art. 90.

224. ***Schol. 2.***—Any locus whose equation contains undetermined constants can in general be made to be tangent to as many given loci as there are independent, undetermined constants in its equation. For, the co-ordinates of the points of intersection of one given locus with this locus can be found by elimination, in terms of the constants of the two equations. If the condition be found that will cause two points of intersection to coincide—that is, form the condition that we have equal roots—the loci will be tangent to each other. This is evidently one equation of condition, and will fix one constant. A tangency with a second locus will fix a second constant, etc.

225. ***E. G.*** Take the circle

$$(x-x_1)^2+(y-y_1)^2=r^2, \quad . \quad . \quad . \quad (a.)$$

containing three independent, undetermined constants, and cause it to touch the three right lines

$$y=x-2, \quad . \quad . \quad . \quad (b.)$$

$$y+x=2, \quad . \quad . \quad . \quad (c.)$$

$$x=4. \quad . \quad . \quad . \quad . \quad . \quad (d.)$$

By eliminating between equations (*a.*) and (*b.*), we obtain

$$x^2-(x_1+y_1+2)\,x+\tfrac{1}{2}(x_1^2+y_1^2+4y_1+4-r^2)=0;$$

and the condition that this quadratic equation in x shall have equal roots is, by algebra,

$$\frac{x_1+y_1+2}{2}=\sqrt{\frac{x_1^2+y_1^2+4y_1+4-r^2}{2}};$$

whence we find

$$(x_1-y_1)^2=2r^2+4x_1-4y_1-4. \quad . \quad . \quad . \quad (e.)$$

which is the equation of condition that the circle touch (*b.*)

Proceeding in a similar manner with (*c.*), we obtain

$$(x_1+y_1)^2=2r^2+4x_1+4y_1-4. \quad . \quad . \quad . \quad (f.)$$

From (*a.*) and (*d.*) we obtain

$$x_1=4\pm r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g.)$$

Equations (*f.*) and (*g.*) are the equations of condition that the circle shall touch (*c.*) and (*d.*) respectively. We have, then, three equations involving three undetermined quantities, x_1, y_1 and r; hence by elimination we can obtain such values as will cause the circle to touch all three lines (*b.*), (*c.*), (*d.*) at once. Subtracting (*e.*) from (*f.*), we obtain

$$x_1\,y_1=2y_1.$$

The roots of this equation are

$$x_1=2 \text{ and } y_1=0.$$

This value of x_1 in (*g.*) gives $r=2$; and placing in equation (*e.*) $x_1=2$ and $r=2$, we find

$$y_1=\pm 2\sqrt{2}.$$

Also if we make $y_1=0$ and $x_1=4\pm r$ in equation (*e.*), we find

$$r=\pm 2\pm 2\sqrt{2}.$$

Since a negative radius gives an imaginary circle, we can only use the upper sign of $\pm 2\sqrt{2}$;

$$\therefore \quad r=\pm 2+2\sqrt{2}=4.828+, \text{ or } 0.828+;$$

whence $$x_1=4\pm(2\sqrt{2}\pm 2)=3.172, \text{ or } 8.828.$$

Thus it appears that there are four circles which touch the three lines, as follows:

1st.	$x_1 = 2,$	$y_1 = +2\sqrt{2},$	$r = 2;$
2d.	$x_1 = 2,$	$y_1 = -2\sqrt{2},$	$r = 2;$
3d.	$x_1 = 6 - 2\sqrt{2},$	$y_1 = 0,$	$r = 2\sqrt{2} - 2;$
4th.	$x_1 = 6 + 2\sqrt{2},$	$y_1 = 0,$	$r = 2\sqrt{2} + 2.$

226. *Examples.*—(1.) Find the equations of condition that the right line $y = mx + b$ may be tangent to the two circles

$$x^2 + y^2 = 4, \quad \text{and} \quad x^2 + y^2 = 12x - 27.$$

$$\textit{Ans. } b = \pm 2\sqrt{1 + m^2}, \quad \text{and} \quad b = -6m \pm 3\sqrt{1 + m^2}.$$

(2.) Find the resulting values of b and m.

$$\textit{Ans. } b = \pm 12\sqrt{\frac{1}{35}}, \text{ or } \mp 12\sqrt{\frac{1}{11}}, \text{ and } m = \pm\sqrt{\frac{1}{35}}, \text{ or } \pm 5\sqrt{\frac{1}{11}}.$$

(3.) Show that the equations of the four lines which touch the two circles are

$$y\sqrt{35} = x + 12, \quad \text{and} \quad y\sqrt{11} = 5x - 12,$$

$$y\sqrt{35} = -x - 12, \quad \text{and} \quad y\sqrt{11} = -5x + 12.$$

Proposition 6.†

227. *Theorem.*—*The general equation of the n^{th} degree—i.e., an equation in which all the constants have general values—can be made to fulfill $\frac{1}{2}n(n + 3)$ conditions.*

For, the general equation of the first degree contains two constants independent of each other, that of the second degree $2 + 3$, and that of the n^{th} degree $2 + 3 + 4 + \ldots\ldots + (n + 1)$, or, by summing the series, $\frac{1}{2}n(n + 3)$ constants; and hence (Art. 221) can fulfill $\frac{1}{2}n(n + 3)$ conditions.

228. ***Schol.***—Some curve of the second degree can be made to pass through any *five* points, or be tangent to any *five* given loci (Arts. 222 and 224).

Proposition 7.

229. ***Theorem.***—*Transformation of co-ordinates cannot change the form of the locus, nor the degree of any equation transformed.*

The *form* of the *locus* is unchanged, for that depends upon the *relation* of the different positions of the moving point to each other, and not upon the *co-ordinates* by which relation is expressed.

The *degree* is unchanged, for in any transformation by equations of the form

$$x = m_1 x' + m_2 y' + m_3$$

$$y = n_1 x' + n_2 y' + n_3$$

(Art. 89) the values of the new variables being of the first degree in terms of the primitive, x^2, or xy, or y^2 could at most be composed only of terms containing x'^2, $x' y'$, x'^2, x', y', etc.; therefore the degree of the expression in the new variables could not be *increased.* Neither could it be *diminished;* for if any transformation could cause the resulting equation to be of *lower* degree, then, by a retransformation to the original expression, an equation of *higher* degree would be obtained, which has been proved to be impossible.

230. ***Example.***—Refer the locus represented by the equation $x^2 + y^2 = 9$, in rectangulars, to new axes in which $\frac{y}{x} = 60°$, making the co-ordinates of the centre $(3, 4)$, thus showing that the degree of the equation is not changed by the transformation.

$$\textit{Ans. } x'^2 + y'^2 + x' y' + 6x' + (3 + 4\sqrt{3}) y' = -16.$$

The equation $x^2 + y^2 = 9$ represents a circle, and the new equation is evidently also that of a circle, since it is of the form of Art. 174.

Proposition 8.†

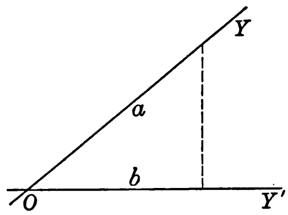
231. Theorem.—*When, in any transformation of co-ordinates, some of the constants are undetermined, the new axes may, in general, be made to fulfill as many conditions as there are independent, undetermined constants in the equations of transformation.*

For, evidently, definite values must in some way be assigned to the undetermined constants, in order to fix the position of the new axes. *One* equation of condition is, in effect, the determination of the value of *one* constant, *two* conditions of *two* constants, etc. (compare Art. 221).

232. Schol.—There are evidently four independent constants in the most general transformation possible; for the axis of x' can be made to pass through any two points, and y' through any other two; or each can be made tangent to two different loci, etc., etc.

Proposition 9.†

233. Theorem.—*Replace* x *by* $\frac{b}{a}x'$, *or* y *by* $\frac{a}{b}y'$, *in any equation of a locus, and the resulting locus will be an orthogonal projection (Art. 60) of the first locus upon some oblique plane.*



For, evidently, if the axis of x be supposed to pass through O, and to be perpendicular to the paper, and the angle YOY' be such that $\frac{a}{b} = sec\ YOY'$; then, if the locus of (x, y) is in the plane XOY, when $y = \frac{a}{b}y'$, or $\frac{y}{y'} = \frac{a}{b}$, the locus of (x, y'), in the plane XOY', will be at the foot of the perpendicular upon XOY' through (x, y). The same may be proved if $x = \frac{b}{a}x'$.

234. ***Schol.***—When ρ is replaced by $\frac{\rho}{m}$ in any equation expressed in polar co-ordinates, the scale of curve represented is thereby changed; *e. g.*, if $m = 2$ the scale is doubled, and if $m = \frac{1}{2}$ the scale is decreased one half.

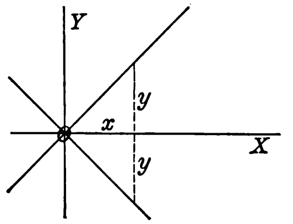
When θ is replaced by $\frac{\theta}{n}$, the loops or other parts of the curve which were contained in any angle θ_1 are caused to be contained in n times that angle; *e. g.*, if $n = \frac{1}{2}$, the part of the figure which was described by any single rotation of the radius vector through *360°* will be completely described during the rotation of the radius vector through *180°*.

This will appear more clearly in the chapter on polar curves and spirals.

Proposition 10.

235. ***Theorem.***—*The general equation of the n^{th} degree represents not only the curves peculiar to that degree, but also all curves represented by equations of inferior degrees.*

For, since in a general equation the constants may have any finite value, among other values are those which will enable us to separate the equation into factors, whose product is of the n^{th} degree, and equal to zero. Since the product $= 0$, by the "General Theory of Equations," each factor $= 0$, and so represents a curve of lower degree.



236. ***Schol.***—The product of two equations of loci represents at once the two loci. *E. G.* Take the two equations $x + y = 0$ and $x - y = 0$, whose product is $x^2 - y^2 = 0$.

$\therefore\ y = \pm\sqrt{x^2} = \pm x$, and for every value of x we have two values of y, equal, but with opposite signs.

It may also be noticed that the product of two equations of loci put equal to some constants represents a locus which approximates some-

what to the two loci. *E. G.* The equation $x^2-y^2=1$ represents a curve approximating in position to the lines $x+y=0$, $x-y=0$ —*i. e.*, $x^2-y^2=0$ (see fig. in Art. 358).

Proposition 11.†

237. Theorem.—*A locus has an axis of symmetry in the following cases:*

***1st.** It is symmetric about the axis of x when its equation is unchanged by a change in the sign of the y co-ordinate—i. e., the equation contains only even powers of y.*

For, then, each value of x corresponds to two points (x, y) and $(x, -y)$ which are situated symmetrically about the axis of x.

E. G. $y^2=ax^3$ (fig. in Art. 393).

***2d.** It is symmetric about the axis of y when its equation involves only even powers of x.*

E. G. $x^2=4py$ (fig. in Art. 245).

***3d.** It is symmetric about both the axes of x and y when its equation involves only even powers of both x and y.*

E. G. $x^2+y^2=r^2$ (Art. 171).

***4th.** It is symmetric about the bisector of the angle $\overset{+}{X}O\overset{+}{Y}$ when its equation is unchanged by interchanging x and y.*

For, to each point (x, y) corresponds a point (y, x) situated symmetrically about this bisector. *E. G.* $xy=m$ (Art. 361).

***5th.** It is symmetric about the bisector of the angle $\overset{-}{X}O\overset{+}{Y}$ when its equation is unchanged by substituting for x and y, $-y$ and $-x$ respectively.*

For, to every point (x, y) corresponds a point $(-y, -x)$ situated symmetrically about this bisector. *E. G.* $xy=m$.

6th. ***It is symmetric about both these bisectors when the equation is unchanged by substituting for x and y, both y and x, and also $-y$ and $-x$ respectively.***

$$E.\,G. \qquad x^2 + xy + y^2 = 1.$$

Proposition 12.†

238. ***Theorem.—1st.*** ***A locus has four parts or branches alike when its equation is unchanged by substituting for x and y either y and $-x$ or $-y$ and x respectively.***

For, on putting instead of x and y, y and $-x$ respectively, every point (x, y) has a corresponding point $(y, -x)$ which bears the same relation to the origin and the axis of y that (x, y) bears to the origin and the axis of x—*i. e.*, one part or branch of the locus rotated 90° in its plane about the origin then coincides with another part or branch. In the same way make another and another rotation, thus showing that the curve is alike in the four angles.

$$E.\,G. \qquad x^4 + xy = y^4.$$

2d. ***A locus has two parts or branches alike when its equation is unchanged by substituting for x and y, $-x$ and $-y$ respectively.***

For two applications of the operation in the previous case is this operation—*i. e.*, a rotation through 180°.

$$E.\,G. \qquad x^2 - y^2 = 1.$$

CHAPTER VI.

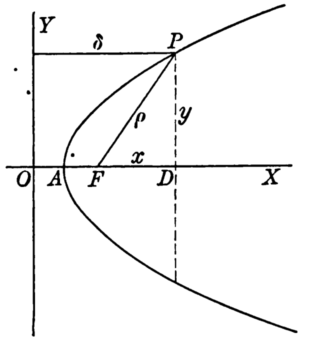
THE PARABOLA.

INTRODUCTORY TO THE CONIC.

239. ***Definition.***—*The equation*

$$\rho = e\delta$$

*represents a **conic section**; in which ρ is the distance of any point of the conic from a given **focus**, and δ is the distance of that point from a given right line, called the **directrix**, and e is any constant called the **eccentricity**.*



The curve is *symmetric* about the axis (see Art. 5). Let the fixed line be the axis of y, and the perpendicular to it through F, the axis of x; then, expressing the above equation in rectangular co-ordinates,

let $OF = 2p$, then $\rho = \sqrt{y^2 + (x - 2p)^2}$,

and $\delta = x$, $\therefore \sqrt{y^2 + (x - 2p)^2} = ex$.

240. $\therefore$ *The equation*

$$y^2 + (x - 2p)^2 = e^2x^2$$

represents a conic referred to a directrix and principal axis.

When $e < 1$, the conic is called **an ellipse.**

When $e = 1$, the conic is called **a parabola.**

When $e > 1$, the conic is called **an hyperbola.**

241. *Exercise.*—The equation

$$(x-x_1)^2+(y-y_1)^2=\frac{e^2(Ax+By+C)^2}{A^2+B^2}$$

is the general form of $\rho=e\delta$ in rectangulars.

242. A **Focal Chord,** or **Parameter,** is any chord of the curve passing through the focus.

The Latus Rectum, or parameter of the principal axis, is parallel to the directrix.

A Diameter is the right line which is the locus of the centres of parallel chords, as will be shown hereafter.

The Centre is the point of intersection of diameters, as will be shown hereafter.

The Vertex is at A, in the figure of Art. 239.

Conjugate Diameters are so situated that each bisects a system of chords parallel to the other.

Supplementary Chords extend from any point of a curve to the extremities of the same diameter.

THE PARABOLA.

Proposition 1.

243. ***Theorem.***—*The equation*

$$y^2=4px$$

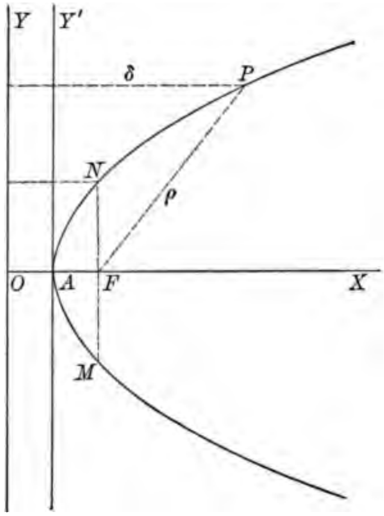
represents a parabola; in which x and y are the rectangular co-ordinates of any point of the curve, and $4p$ is the latus rectum, when the origin is at the vertex, and the axis of x is the axis of the curve.

For, if (Art. 240) $e=1$, then $y^2+(x-2p)^2=x^2$.

Reducing,

$$y^2=4p\,(x-p)\ldots(a.)$$

Let P coincide with A; then $y=0$. $\therefore$ $OA=x=p$. But (Art. 239) if $e=1$, then $\rho=\delta$. $\therefore$ $OA=AF$; $\therefore$ the vertex of a parabola bisects the distance between the directrix and focus. Move the origin to A, then



$$x_0=p, \text{ and } y_0=0;$$

$$\therefore \text{ (Art. 23) } x=p+x',$$

$$\text{and} \qquad y=y'.$$

Substituting these values of x and y in (*a.*), we have,

$$y'^2=4px', \text{ or } y'=\pm 2\sqrt{px'}.$$

Omit the primes: $\therefore$ $y^2=4px$.

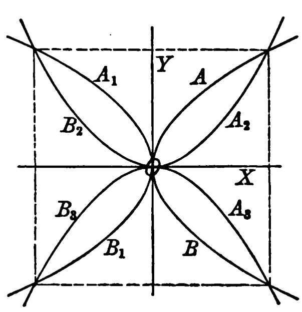
244. *Cor. 1.*—If the origin be moved any distance $x_0=\pm a$ on the axis of x, the equation becomes (Art. 23) $y^2=4p\,(x\mp a)$.

245. *Cor. 2.*—The equation $y^2=4px$ represents a parabola in the position AOB, with the axis of y tangent at the vertex.

$y^2=-4px$ represents A_1OB_1, with axis of y tangent at the vertex.

$x^2=4py$ represents A_2OB_2, with the axis of x tangent at the vertex.

$x^2=-4py$ represents A_3OB_3, with the axis of x tangent at the vertex.



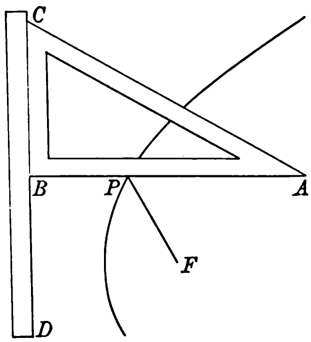
246. ***Cor. 3.***—If in the equation $y^2 = 4px$ we let $x = p$, then $y = \pm 2p = \frac{1}{2}$ (*latus rectum*): and (Art. 243) the distance from the vertex to the focus, or from the directrix to the vertex, is $p = \frac{1}{4}$ (*latus rectum*).

Also, if $y^2 = 4px$, then $x : y :: y : 4p$, $\therefore$ in the parabola the latus rectum is a *third proportional* to the abscissa and ordinate.

If (x_1, y_1) and (x_2, y_2) are upon the parabola, then $y_1^2 = 4px_1$, and $y_2^2 = 4px_2$, $\therefore$ $x_1 : x_2 :: y_1^2 : y_2^2$.

247. ***Schol. 1.***—Since $\rho = \delta = x' + p$, omit the prime, and the equation $\rho = x + p$ is called the *linear* equation* of the parabola. By it the parabola may be constructed.

248. ***Schol. 2.***—The equation $\rho = \delta$ enables us to construct the parabola by continuous motion, with the aid of a ruler CD, a triangle ABC, and a thread equal to AB fastened at A and F. If a pencil P be kept in contact with the triangle along AB, and the triangle slide along the ruler, evidently $PB = PF$, and the locus of P is a parabola.



Proposition 2.

249. ***Theorem.***—*The equation*

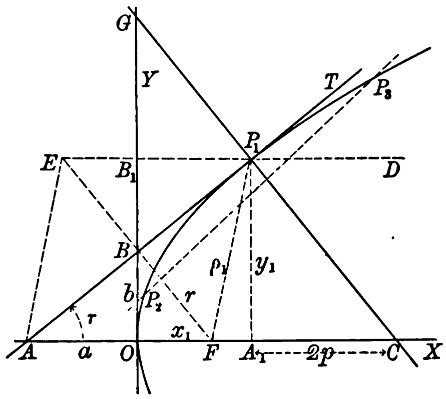
$$yy_1 = 2p\,(x + x_1)$$

represents a line tangent to the parabola $y^2 = 4px$ *at the point* (x_1, y_1) *of the curve.*

* An equation of the first degree is frequently called a *linear equation.*

For (Art. 90),

$$\frac{y-y_3}{x-x_3}=\frac{y_2-y_3}{x_2-x_3} \dots (b.)$$



represents a right line through two points P_2 and P_3. If these points are upon the parabola, the equations of condition which must hold are $y_3^2=4px_3$,

and $y_2^2=4px_2$.

Subtracting, $$y_2^2-y_3^2=4p(x_2-x_3),$$

$$\therefore \quad (y_2-y_3)(y_2+y_3)=4p(x_2-x_3),$$

$$\therefore \quad \frac{y_2-y_3}{x_2-x_3}=\frac{4p}{y_2+y_3},$$

$\therefore$ substituting in (b.) $$\frac{y-y_3}{x-x_3}=\frac{4p}{y_2+y_3}. \dots (c.)$$

If P_2 and P_3 coincide at P_1, then $y_2=y_3=y_1$, and (c.) becomes,

$$y-y_1=\frac{2p}{y_1}(x-x_1),$$

which is the equation of the tangent line. From this

$$yy_1-y_1^2=2px-2px_1; \text{ but } y_1^2=4px_1, \quad \therefore \; yy_1=2p(x+x_1).$$

250. ***Cor 1.***—At the point of contact the equation becomes $y_1^2=4px_1$, as it should, since the point P_1 is on the parabola and on line also.

251. ***Cor. 2.***—If $y=0$, then $2p(x+x_1)=0$, $\therefore \; x=-x_1$, $\therefore$ the intercept $a=x_1$; *i.e.*, $AO=OA_1$, and AA_1, which is called the **subtangent**, is bisected at the vertex.

Also (from sim. tri's) $AO:OA_1::AB:BP_1$, $\therefore \; AB=BP_1$.

252. *Cor. 3.*—If $y = 0$, then, $a = x$.

$\therefore$ (Art. 247), $\rho_1 = p + x_1 = p + a$. —*i.e.*, $AF = FP_1$,

$\therefore$ AFP_1 is an isosceles triangle, and $FAP_1 = AP_1F$.

But if ED is parallel to AX, then $FAP_1 = EP_1A$.

$\therefore$ the tangent AT bisects EP_1F, and $EAFP_1$ is a rhombus whose diagonals bisect each other at right angles on the axis of y.

Again, from the figure, $DP_1T = EP_1A$. $\therefore$ $AP_1F = DP_1T$.

If CP_1 is the normal at P_1, then $AP_1C = CP_1T = 90°$.

Subtract $\therefore$ $AP_1C - AP_1F = CP_1T - DP_1T$.

$\therefore$ $FP_1C = CP_1D$.

Also (from sim. tri's) $AB : BP_1 :: AF : FC$,

$\therefore$ (Art. 251) $AF = FC = \rho_1$.

$\therefore$ CFP_1 is an isosceles triangle, and $FP_1C = P_1CF$.

253. *Cor. 4.*—If $x = 0$, then $b = OB = y = \dfrac{2px_1}{y_1}$.

But $y_1 = 2\sqrt{px_1}$. $\therefore$ $b = \sqrt{px_1}$,

and $AB = \sqrt{a^2 + b^2} = \sqrt{x_1^2 + px_1} = \sqrt{\rho_1 x_1}$ (Art. 247).

Again $a : b :: \sqrt{a^2 + b^2} : r, (= FB)$

or $x_1 : \sqrt{px_1} :: \sqrt{\rho_1 x_1} : r$. $\therefore$ $r = \sqrt{\rho_1 p}$.

Also from (sim. tri's) it may be shown that

$$A_1C = 2OF = 2p,$$

and that $$CP_1 = 2BF = 2\sqrt{\rho_1 p}.$$

254. *Cor. 5.*—From the figure

$$\frac{2p}{y_1} = \tan\frac{t}{x} = \tan\tau.$$

By trig. $$\sin^2\tau = \frac{\tan^2\tau}{1 + \tan^2\tau},$$

$$\sin^2\tau = \frac{\dfrac{4p^2}{y_1^2}}{1+\dfrac{4p^2}{y_1^2}} = \frac{4p^2}{y_1^2+4p^2}.$$

But $\quad y_1^2 = 4px_1 \qquad \therefore\ \sin^2\tau = \dfrac{4p^2}{4px_1+4p^2} = \dfrac{p}{x_1+p}.$

But $\quad \rho_1 = p + x_1;\ \therefore\ \sin^2\tau = \dfrac{p}{\rho_1};\ \therefore\ \rho_1 = \dfrac{p}{\sin^2\tau},$

a form of the linear equation of the parabola of some importance; in which ρ_1 is the radius vector of P_1, and τ the angle between the tangent at P_1 and the axis of x.

Also since $\qquad \dfrac{2p}{y_1} = \tan\tau;$

$$\therefore\ \frac{2p}{y_1} = \frac{\sin\tau}{\cos\tau};\ \therefore\ 2p\cos\tau - y_1\sin\tau = 0.$$

255. ***Examples.***—Construct the tangents represented by the following equations, with the parabolas to which they are tangent, finding the equations of the parabolas, when $y_1 = 4$.

(1.) $\quad y = 4(x+\frac{1}{2}).$ *Ans.* $y^2 = 32x.$

(2.) $\quad y = x + 2.$ *Ans.* $y^2 = 8x.$

(3.) Find the equation of the tangent to the parabola $y^2 = 16x$, when $x_1 = 9$. *Ans.* $3y = 2x + 18.$

Proposition 3.†

256. ***Theorem.***—*The equation*

$$y = mx + \frac{p}{m}$$

also represents a line tangent to the parabola $y^2 = 4px$; *in which*

$$m = \frac{2p}{y_1} = \tan\frac{t}{x}.$$

For, if (Art. 249) $yy_1 = 2p(x + x_1)$ and $y_1^2 = 4px_1$, or $x_1 = \frac{y_1^2}{4p}$,

then, by substitution, $yy_1 = 2p\left(x + \frac{y_1^2}{4p}\right)$;

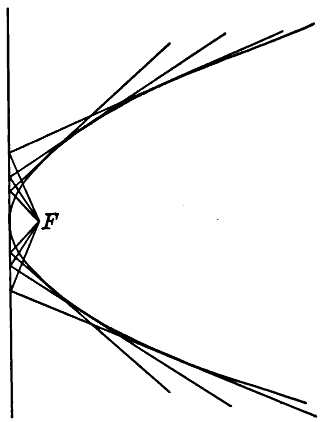
$$\therefore \quad y = \left(\frac{2p}{y_1}\right)x + \left(\frac{y_1}{2p}\right)p, \qquad \therefore \quad y = mx + \frac{p}{m}.$$

This is called the *magic equation* of the tangent line.

257. ***Schol. 1.***—The locus of the foot of the perpendicular from the focus upon the tangent line is the tangent at the vertex. For, the equation of the tangent may be written $my - m^2x = p$, and the perpendicular to it through the focus, $(p, 0)$, is (Art. 128) $my + x = p$.

Subtract, $\therefore \quad (1 + m^2)x = 0$

$\therefore \quad x = 0$, which is the equation of the tangent at the vertex.



This enables us to construct a parabola approximately by drawing successive tangents, as in the figure.

258. ***Schol. 2.***—The locus of the intersections of tangents perpendicular to each other is the directrix.

For, $\quad y = mx + \frac{p}{m}$

represents some tangent; in which equation let us replace m by $\frac{-1}{m}$ (Art. 123), $\therefore \quad y = -\frac{x}{m} - mp$ represents a second tangent perpendicular to the first. Subtract, etc., $\therefore \quad x = -p$, which is the equation of the directrix.

259. ***Exercise.***—Prove that the equation $y = m_0x + \frac{p}{m_0}$ is the locus of the intersection of the tangent with a line drawn from the focus, and making with the tangent line an angle whose tangent $= m_0$.

Proposition 4.

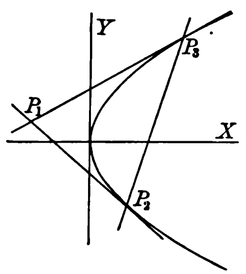
260. ***Theorem.***—*The equation*

$$yy_1 = 2p\,(x + x_1)$$

also represents a right line which is the chord of contact of two tangents drawn to the parabola $y_2 = 4px$ ***from the external point*** (x_1, y_1).

For, let P_2P_3 be the chord of contact of the tangents P_1P_2 and P_1P_3.

From Art. 90, $\dfrac{y - y_3}{x - x_3} = \dfrac{y_2 - y_3}{x_2 - x_3} \ldots$ (*d.*) represents P_2P_3.



From Art. 249, $yy_2 = 2p\,(x + x_2) \ldots$ (*e.*) represents the tangent at P_2, and if it passes through P_1, the co-ordinates of P_1 must satisfy (*e.*).

$$\therefore \quad y_1y_2 = 2p\,(x_1 + x_2) \ldots (f.)$$

similarly $\quad y_1y_3 = 2p\,(x_1 + x_3) \ldots (g.)$

subtract $\therefore \quad y_1(y_2 - y_3) = 2p\,(x_2 - x_3)$

$\therefore \quad \dfrac{y_2 - y_3}{x_2 - x_3} = \dfrac{2p}{y_1}$, substitute in (*d.*). $\therefore \quad \dfrac{y - y_3}{x - x_3} = \dfrac{2p}{y_1}$

clear of fractions. $\therefore \quad yy_1 - y_1y_3 = 2px - 2px_3$.

Add (*g.*). $\therefore \quad yy_1 = 2p\,(x + x_1)$ represents P_2P_3.

P_1 is called the **pole**, and P_2P_3 its **polar**, with respect to the parabola.

261. ***Exercise.***—If the pole is on the *directrix*, show that the polar is a focal chord.

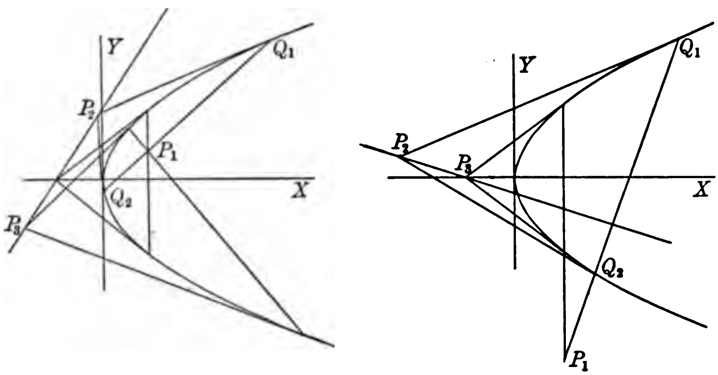
Proposition 5.

262. ***Theorem***—*The equation*

$$yy_1 = 2p\,(x + x_1)$$

also represents a right line which is the locus of the intersections of the pairs of tangents to the parabola $y^2 = 4px$, ***drawn from the extremities of all chords which pass through any fixed point*** (x_1, y_1).

For, let P_1 be the fixed point through which all the chords pass, and let Q_1Q_2 be any chord through P_1. If the tangents at



Q_1 and Q_2 meet in some point P_2, then (Art. 260) $yy_2 = 2p\,(x + x_2)$ is the equation of the chord Q_1Q_2; and at P_1 this equation becomes $$y_1y_2 = 2p\,(x_1 + x_2) \ldots (h.)$$

Similarly, if the tangents at the extremity of another chord through P_1 intersect at P_3, $y_1y_3 = 2p\,(x_1 + x_3) \ldots (k.)$

But (Art. 90), $$\frac{y - y_3}{x - x_3} = \frac{y_2 - y_3}{x_2 - x_3} \ldots (i.)$$ represents P_2P_3.

Eliminate as in Art. 260. $\therefore\ yy_1 = 2p\,(x + x_1)$ represents P_2P_3.

P_1 is called the **pole**, and P_2P_3 its **polar**, with respect to the parabola.

263. ***Schol.***—The tangent (Art. 249) $yy_1 = 2p\,(x + x_1)$ is the particular case in which the pole is upon the curve, and consequently upon its own polar.

264. ***Examples.***—(1.) Given a parabola whose latus rectum is 8, to find the polar of the point $(3, 7)$. *Ans.* $7y = 4x + 12$.

(2.) The latus rectum of a parabola is 4; find the pole of the line $y = 8x + 4$. *Ans.* $x_1 = \frac{1}{2}$, $y_1 = \frac{1}{4}$.

(3.) Given a parabola $y^2 = x$, and a point $(-4, 10)$, to find the intercepts of its polar. *Ans.* $a = 4$, $b = -\frac{1}{5}$.

265. ***Exercise.***—If the *focus* is the pole, the *directrix* is the polar.

Proposition 6.

266. ***Theorem.***—*The equation*

$$y - y_1 = -\frac{y_1}{2p}(x - x_1)$$

is that of a line normal to the parabola $y^2 = 4px$ *at the point* (x_1, y_1).

For, resuming the equation of the tangent line (Art. 249),

$$y - y_1 = \frac{2p}{y_1}(x - x_1),$$

we have (Art. 123), $\quad y - y_1 = -\frac{y_1}{2p}(x - x_1),$

perpendicular to the tangent, and it passes through the point of contact (x_1, y_1) by Art. 98.

267. ***Schol. 1.***—The equation $y - y_1 = -\frac{y_1}{2p}(x - x_1)$ also represents the normal when the origin is at *any* point on the axis of x. Move the origin so that the axis of y shall pass through the point (x_1, y_1)—*i. e.*, let $x_1 = 0$; then $\frac{y}{y_1} + \frac{x}{2p} = 1$, and (figure of Art. 249) $A_1C = 2p \quad \therefore$ **subnormal** $=$ a constant.

268. ***Schol. 2.***†—Let $m = -\frac{y_1}{2p} = \tan\frac{l}{x}, \quad \therefore \quad y_1 = -2pm.$

Also, since $y_1^2 = 4px_1, \quad \therefore \quad x_1 = p\left(\frac{y_1}{2p}\right)^2 = pm^2.$

Substitute these values in $y - y_1 = -\frac{y_1}{2p}(x - x_1),$

then, $\quad y + 2pm = m(x - pm^2)$

or, $\quad y = mx - pm(2 + m^2),$

which is the magic equation of the normal to the parabola.

269. *Examples.*—(1.) Find the equation of the normal to the parabola, when $p = 2$ and $x_1 = 32$. *Ans.* $y = -4x + 144$.

(2.) Find the equation of the normal to the same parabola, passing through the point $(10, 4)$ not on the curve, the equation of condition that the normal shall pass through a point (x_2, y_2) being (Art. 266)

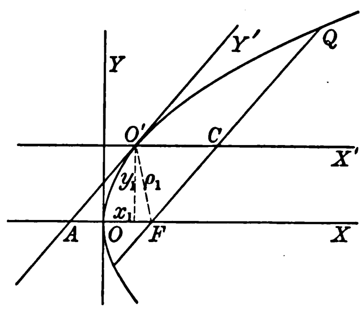
$$\frac{y_1 - y_2}{x_1 - x_2} = -\frac{y_1}{2p}.$$ *Ans.* $y + 2x = 24$ is normal at $(8, 8)$.

Proposition 7.

270. *Theorem.*—*The equation*

$$y^2 = 4p'x$$

represents a parabola referred to a tangent, and a parallel to the axis through the point of contact; in which p' is the distance from the focus to the point of contact, and $4p'$ is the length of the focal chord parallel to the tangent.



For, transfer the origin from the vertex O to some point O' upon the parabola $y^2 = 4px. \; . \; . \; .$ (*I.*) The equations of transformation are (Art. 89, 3),

$$x = x_1 + x' + y' \cos \tau$$

$$y = y_1 + y' \sin \tau$$

when we make the axis of x' parallel to the axis of x, and the axis of y' the tangent at O'. Therefore, substituting these in (*I.*),

$$(y_1 + y' \sin \tau)^2 = 4p\,(x_1 + x' + y' \cos \tau),$$

or expanding and arranging,

$$y'^2 \sin^2 \tau + y_1^2 - 4px_1 = 4px' + 2y'\,(2p \cos \tau - y_1 \sin \tau).$$

But, since (x_1, y_1) is on the curve, $y_1^2 - 4px_1 = 0$,

and (Art. 254), $\quad 2p \cos \tau - y_1 \sin \tau = 0$,

hence $\quad y'^2 = \dfrac{4p}{\sin^2 \tau} x'$; $\quad \therefore$ (Art. 254), $y'^2 = 4\rho_1 x'$.

Again (Art. 252), $\quad \rho_1 = AF = O'C$.

Let $x' = \rho_1 = O'C$. $\quad \therefore \; y'^2 = 4\rho_1^2$. $\quad \therefore \; y' = \pm 2\rho_1 = CQ$.

If, for convenience, $\rho_1 = p'$, then, $4\rho_1 = 4p'$ is the length of the focal chord parallel to the axis of y'.

Hence, omitting the primes, $y^2 = 4p'x$.

271. *Cor.*—Since $y = \pm 2\sqrt{p'x}$, every positive value of x gives two equal values of y, or the axis of x is the bisector of all chords parallel to the tangent, and is therefore a *diameter*, and all lines parallel to the principal axis are diameters.

272. *Schol.*—Since parallel chords all pass through the same point (∞_1, ∞_2) at infinity, and since for this point $x_1 = \infty_1$ and $y_1 = \infty_2$, the equation of the polar of this point (Art. 262) $yy_1 = 2p\,(x + x_1)$ or, $y = \dfrac{2px}{y_1} + \dfrac{2px_1}{y_1}$ becomes $y = \dfrac{2px_1}{y_1}$, or $y =$ a constant. This polar of parallel chords is then (Art. 106) parallel to x, and cuts parabola at the point of contact of the tangent parallel to the chords. Hence the locus of the intersections of pairs of tangents at the extremities of parallel chords is the diameter bisecting those chords.

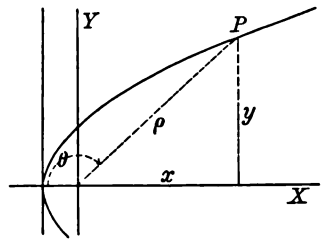
Proposition 8.

273. *Theorem.*—*The equation*

$$\rho = \frac{2p}{1 + \cos \theta}, \quad \text{or} \quad \rho = \frac{l}{1 + \cos \theta}$$

represents a parabola; in which ρ is the radius vector, and θ the variable angle of any point of the curve, measured from the vertex, when the pole is at the focus, and $l = 2p$ is the semi-latus rectum.

For, $y^2 = 4p(x+p)$ is the equation of a parabola referred to the rectangular axes through the focus. Transform by the equations



$x = -\rho \cos\theta$ and $y = \rho \sin\theta$.

$\therefore \quad \rho^2 \sin^2\theta = -4p\,\rho\cos\theta + 4p^2$

$$\therefore \quad \rho = \frac{2p\,(\pm 1 - \cos\theta)}{\sin^2\theta} = \frac{2p\,(\pm 1 - \cos\theta)}{1 - \cos^2\theta}.$$

Cancel $\quad \therefore \quad \rho = \dfrac{2p}{1+\cos\theta}, \quad$ and $\quad -\rho = \dfrac{2p}{1-\cos\theta},$

which are the positive and negative values of ρ corresponding to the angle θ.

274. ***Schol.***—By trig.,

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}, \text{ and } 1 - \cos\theta = 2\sin^2\frac{\theta}{2}.$$

$$\rho = \frac{2p}{1+\cos\theta} = \frac{p}{\cos^2\frac{\theta}{2}}, \quad \text{and} \quad -\rho = \frac{p}{\sin^2\frac{\theta}{2}}.$$

275. ***Exercise.***—Prove that

$$\rho = \frac{4p\cos\theta}{1-\cos^2\theta}$$

is the polar equation of the parabola when the pole is at the vertex.

Proposition 9.†

276. ***Theorem.***—*The equation*

$$\rho = \frac{l}{\cos(\theta - \theta_1) + \cos\theta}$$

represents a line tangent to the parabola $\rho = \dfrac{l}{1+\cos\theta}$ *at the point* (ρ_1, θ_1), *the pole being at the focus.*

For, $yy_1 = 2p(x + x_1 + 2p)$ is the equation of the tangent line referred to rectangular axes through the focus. Transform by equations

$$x = -\rho\cos\theta, \text{ and } y = \rho\sin\theta;$$

$$\therefore \quad \rho\rho_1 \sin\theta \ \sin\theta_1 = -2p\,\rho\cos\theta - 2p\,\rho_1\cos\theta_1 + 4p^2.$$

But $\qquad \rho_1 = \dfrac{2p}{1+\cos\theta_1}.$

Substitute, clear of fractions, transpose and solve for ρ,

$$\therefore \quad \rho = \frac{2p}{\cos(\theta-\theta_1)+\cos\theta} = \frac{l}{\cos(\theta-\theta_1)+\cos\theta}.$$

277. *Schol. 1.*—If $\theta = \theta_1$, this becomes the equation of the curve, as it should.

278. *Schol. 2.*—By trigonometry,

$$\rho = \frac{l}{\cos^2(\theta-\theta_1)+\cos\theta} = \frac{p}{\cos\left(\theta-\frac{\theta_1}{2}\right)\cos\frac{\theta_1}{2}}.$$

279. *Examples.*—(1.) Construct the tangent to the parabola $\rho = \dfrac{4}{1+\cos\theta}$ at the point whose vectorial angle is $\theta_1 = 60°$, and find the angle which it makes with the initial line.

Ans. The angle $= 60°$.

(2.) Find from (Art. 280) the normal to the same parabola at the same point.

280. *Exercise.*—Prove that the equation of the normal, when the pole is at the focus, is

$$\rho = \frac{p\sin\frac{\theta_1}{2}}{\cos^2\frac{\theta_1}{2}\sin\left(\theta-\frac{\theta_1}{2}\right)}.$$

If $\theta = \theta_1$, this also becomes the equation of the curve.

CHAPTER VII.

HYPERBOLA AND ELLIPSE.

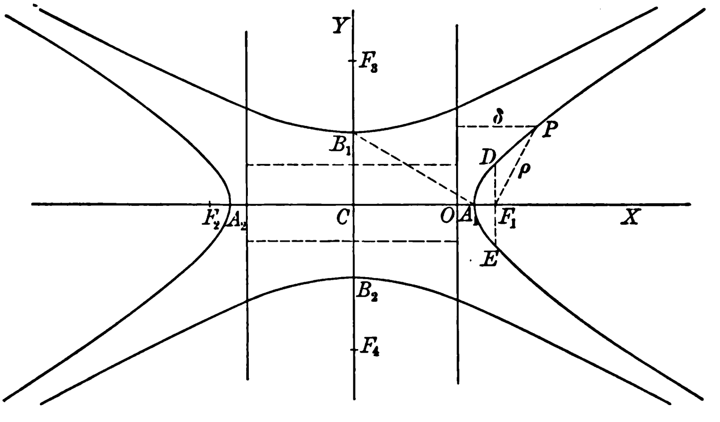
Proposition 1.

281. ***Theorem.***—*The equation*

$$\frac{x^2}{a^2}+\frac{y^2}{-b^2}=1$$

represents an hyperbola with the origin at the centre; in which a^2 ***and*** $-b^2$ ***are the squares of the intercepts on*** x ***and*** y ***respectively—i. e., the semi-axes.***

For, $y^2+(x-2p)^2=e^2x^2$. . . (*a.*) is by definition (Art. 240)



the equation of an hyperbola when $e>1$, with the origin at O.

If $\qquad y=0,\quad$ then $\quad x-2p=\pm\, ex,$

and $\qquad \therefore\ x_0=\dfrac{2p}{1+e}$ is the intercept OA_1, and $x_0=\dfrac{2p}{1-e}$ is the intercept OA_2.

Let $\qquad CA_1 = +a, \quad \text{and} \quad CA_2 = -a,$

then (Art. 9), $CA_1 = \frac{1}{2}(OA_1 - OA_2) = \frac{p}{1+e} - \frac{p}{1-e} = -\frac{2pe}{1-e^2} = a.$

$$\therefore \quad 2p = -\frac{a}{e}(1-e^2).$$

Again, $OC = \frac{1}{2}(OA_1 + OA_2) = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2} = -\frac{a}{e}.$

Move origin to C (Art. 23), $\therefore \quad y = y'$, and $x = x' - \frac{a}{e}$,
and substituting these in (*a.*),

$$\therefore \quad y'^2 + [x' - \frac{a}{e} + \frac{a}{e}(1-e^2)]^2 = e^2\left(x' - \frac{a}{e}\right)^2$$

$$\therefore \quad y'^2 + x'^2 + a^2e^2 = e^2x'^2 + a^2,$$

$$\therefore \quad y'^2 = a^2 - x'^2 - e^2(a^2 - x'^2) = (1-e^2)(a^2 - x'^2).$$

If $\qquad x' = 0, \text{ then } y'^2 = a^2(1-e^2),$

the second member of which is negative when $e > 1$,

$$\therefore \text{ let } \quad a^2(1-e^2) = -b^2;$$

then, $\qquad y'^2 = \frac{-b^2}{a^2}(a^2 - x'^2), \text{ or } \frac{x^2}{a^2} + \frac{y^2}{-b^2} = 1, \ldots (b.)$

(dropping the primes) represents the same curve that (*a.*) does.

282. ***Cor. 1.***—This curve is in form symmetrical about both the axes of x and y, for every value of x gives two values of y numerically equal, but of opposite signs, and vice versa.

There are no *real* values of y in this curve between $x = +a$ and $x = -a$, as may be seen from the equation, when put in the form,

$$y = \pm\frac{a}{b}\sqrt{x^2 - a^2}.$$

283. ***Cor. 2.***—The latus rectum $ED=\dfrac{2b^2}{a}$.

For, if $\quad CF_1=OF_1-OC=2p-\left(-\dfrac{a}{e}\right)=-\dfrac{a}{e}(1-e^2)+\dfrac{a}{e}=ae,$

let $\quad x_f=ae;\quad \therefore\quad y_f^2=\dfrac{-b^2}{a^2}(a^2-a^2e^2)=\dfrac{b^4}{a^2}.\quad \therefore\quad 2y_f=\dfrac{2b^2}{a}.$

284. ***Cor. 3.***—Since $\quad a^2(1-e^2)=-b^2,$ or $a^2+b^2=a^2e^2,$

$$\therefore\quad e=\frac{\sqrt{a^2+b^2}}{a},\quad \therefore\quad \text{also } e=\frac{CF_1}{CA_1}=\frac{A_1B_1}{CA_1}=\sec CA_1B_1,$$

for, (Art. 283) $CF_1=ae=\sqrt{a^2+b^2}$, and from the fig. $A_1B_1=\sqrt{a^2+b^2}$.

$$\text{Also,}\qquad CO=\frac{a}{e}=\frac{a^2}{\sqrt{(a^2+b^2)}}.$$

285. ***Schol. 1.***—The axis $A_1A_2=2a$ is called the **transverse axis.** The axis $B_1B_2=2b$ at right angles to the transverse axis is the **conjugate axis.**

286. ***Schol. 2.***—The hyperbola becomes *equilateral* or *rectangular* when the axes are of equal magnitude;

$x^2-y^2=a^2$ is the equation of an equilateral hyperbola.

287. ***Examples.***—Find the value of CF_1, CA_1, CB_1 and ED for the hyperbola represented by each of the following equations, and construct their directrices, vertices, foci and the extremities of each latus rectum.

(1.) $\quad \dfrac{x^2}{9}+\dfrac{y^2}{-4}=1.\qquad$ *Ans.* $a=\pm 3,\ b=\pm 2\sqrt{-1}.$

(2.) $\quad \dfrac{4x^2}{9}+\dfrac{y^2}{-9}=1.\qquad$ *Ans.* $a=\pm 1\tfrac{1}{2},\ b=\pm 3\sqrt{-1}.$

(3.) $\quad \dfrac{4x^2}{25}-y^2=1.\qquad$ *Ans.* $a=\pm 2\tfrac{1}{2},\ b=\pm\sqrt{-1}.$

(4.) Show that the latus rectum is a third proportional to the axes $2a$ and $2b$.

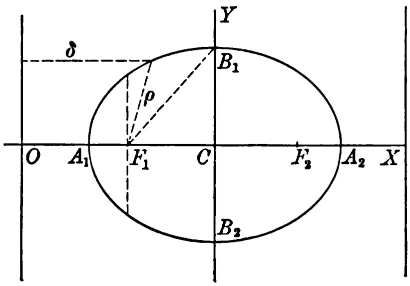
(5.) Show that a is mean proportional between CF_1 and CO.

Proposition 2.

288. ***Theorem.**—The equation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

represents an ellipse with the origin at the centre; in which a^2 and b^2 are the squares of the semi-axes, and $a > b$.



The demonstration of this proposition is identical with the preceding, with this exception, that

$$a^2(1-e^2)=+b^2,$$

since (Art. 240) $e < 1$.

The lettering of Art. 281 will be found to apply to this figure, and the results will therefore agree with those in Art. 281 on changing the sign of b^2.

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad . \quad . \quad . \quad (c.)$$

289. ***Cor. 1.***—The curve is *symmetrical* about the axes of x and y. The values of y are all *real* between $x=+a$, and $x=-a$, but *imaginary* beyond, since $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$.

290. ***Cor. 2.***—The latus rectum $=\frac{2b^2}{a}$ (Art. 283).

291. ***Cor. 3.***—Since $a^2(1-e^2)=b^2$, $\therefore \quad e=\frac{\sqrt{a^2-b^2}}{a}$.

$$\therefore \quad F_1C=ae=\sqrt{a^2-b^2}, \quad \text{and} \quad F_1B_1^2=F_1C^2+CB_1^2,$$

$$\therefore \quad F_1B_1^2=(a^2-b^2)+b^2=a^2, \quad \therefore \quad A_1C=F_1B_1=a.$$

$$\therefore \quad e=\frac{F_1C}{A_1C}=\frac{F_1B_1}{A_1C}=\cos CF_1B_1. \quad \text{Also } OC=\frac{a}{e}=\frac{a^2}{\sqrt{(a^2-b^2)}}.$$

292. ***Schol. 1.***—The axis $A_1A_2=2a$ is the *transverse* or ***major axis,*** and $B_1B_2=2b$ is the *conjugate* or ***minor axis.***

293. ***Schol. 2.***—The ellipse becomes a circle when $a^2 = b^2 = r^2$. Then $e = 0$, and the equation becomes $x^2 + y^2 = r^2$.

294. ***Schol. 3.***—When $b > a$, then $e_1 = \frac{\sqrt{a^2 - b^2}}{a}$ is imaginary, but $e_2 = \frac{\sqrt{b^2 - a^2}}{b}$ is real, $\therefore$ b is the major and a is the minor axis.

295. ***Schol. 4.***—The equations of the ellipse and hyperbola may be written together,

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1 \quad . \; . \; . \; (\textit{bc.})$$

in which the sign $+$ is for the ellipse and $-$ for the hyperbola.

296. ***Schol. 5.***—The equation $\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{\pm b^2} = 1$ represents an ellipse or hyperbola whose axes of reference are parallel to its axes of figure; in which x_1 and y_1 are the co-ordinates of the centre. This may be shown by moving the origin (Art. 23) in the equation $\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1$. If $x_1 = \pm a$, then the origin is at A_1 or A_2, and we have for the equation at the vertex $\frac{x^2 \mp 2ax}{a^2} + \frac{y^2}{\pm b^2} = 0$,

$$\therefore \quad y^2 = -\frac{\pm b^2}{a^2}(x^2 \mp 2ax), \text{ or } y^2 = \pm \frac{2b^2}{a} x \mp \frac{b^2}{a^2} x^2.$$

297. ***Schol. 6.***—The equation of the ellipse is (Art. 289), $\frac{a}{b} y_e = \sqrt{a^2 - x^2}$, and that of the circle (Art. 171), $y_c = \sqrt{a^2 - x^2}$. For any value of x common to the ellipse and the circle we have,

$\therefore \quad y_c = \frac{a}{b} y_e$, or $\frac{y_c}{y_e} = \frac{a}{b}$. $\therefore$ by Art. 233 the ellipse is a projection of the circle when the major axis is the line common to both. The same may be proved of the circle on its minor axis—*i. e.*, for any value of y common to both, $\frac{x_c}{x_e} = \frac{b}{a}$.

298. ***Examples.***—(1.) Change the sign of the term containing y^2 in each of the equations of Art. 287 from — to +, and find the position of the foci, etc., of the ellipse represented by each equation.

(2.) Show that the proportions stated in Art. 287, (4) and (5), are true in case of the ellipse.

Proposition 3.

299. ***Theorem.***—*The equation*

$$\frac{x^2}{-a^2}+\frac{y^2}{b^2}=1$$

represents an hyperbola conjugate to

$$\frac{x^2}{a^2}+\frac{y^2}{-b^2}=1$$

—i. e., the transverse axis of the one is the conjugate axis of the other, and vice versa.

For, evidently this represents the curve whose vertices are at B_1 and B_2 (Art. 281), and it may be considered to be derived from

$$\frac{x^2}{a^2}+\frac{y^2}{-b^2}=1$$

by the exchange of $\frac{x}{a}$ and $\pm\frac{y}{b}$. Also since $-a^2$ is the square of a semi-axis, that axis is imaginary, while the other is real.

Similar corollaries and scholia apply to this proposition and Proposition 1.

300. ***Cor.***—In the conjugate hyperbola

$$e=\frac{\sqrt{a^2+b^2}}{b}=\frac{A_1B_1}{CB_1}=\sec CB_1A_1,$$

for, $\quad CF_3=A_1B_1=CF_1=\sqrt{a^2+b^2}$, and $CB_1=b$.

Also, the distance from C to the directrix $=\frac{b}{e}=\frac{b^2}{\sqrt{(a^2+b^2)}}$.

Proposition 4.

301. ***Theorem.***—*The equation*

$$\frac{x^2}{-a^2}+\frac{y^2}{-b^2}=1$$

represents an imaginary ellipse.

For, from this equation $y=\pm\frac{b}{a}\sqrt{-a^2-x^2}$, hence all values of x give imaginary values of y; still either e_1 or e_2 is real, and has the same value as in the real ellipse (Art. 294).

Proposition 5.

302. ***Theorem.***—*The equations*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2\pm d^2}+\frac{y^2}{b^2\pm d^2}=1$$

represent conics having the same foci, and are called confocal conics.

For, in the first curve (Art. 288) $a^2e_1^{\,2}=a^2-b^2$, and in the second

$$(a^2\pm d^2)e_2^{\,2}=a^2\pm d^2-(b^2\pm d^2)=a^2-b^2;$$

$\therefore$ the distance from the centre to the focus in any conic represented by the equation

$$\frac{x^2}{a^2\pm d^2}+\frac{y^2}{b^2\pm d^2}=1 \ldots (\boldsymbol{d.})$$

is $\sqrt{a^2-b^2}$, whatever value d^2 may have.

This is true whether b is real or imaginary.

303. ***Examples.***—(1.) Find the nature of the curves represented by the equation (***d.***) when $a^2=4$, $b^2=1$, and $d^2=$ successively 2, 1, 0, with the upper sign, and 1, 2, 3, 4, with the lower sign.

(2.) Find the vertices, etc., of the hyperbolas conjugate to those of Art. 287.

Proposition 6.

304. ***Theorem.***—*The equations*

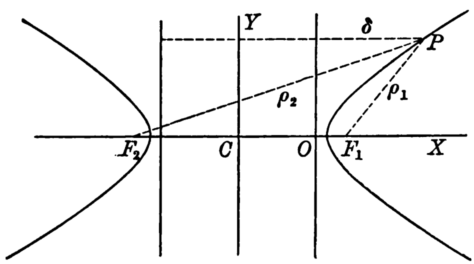
$$\rho_1 = ex - a \quad \text{and} \quad \rho_2 = ex + a$$

also represent the hyperbola

$$\frac{x^2}{a^2} + \frac{y^2}{-b^2} = 1;$$

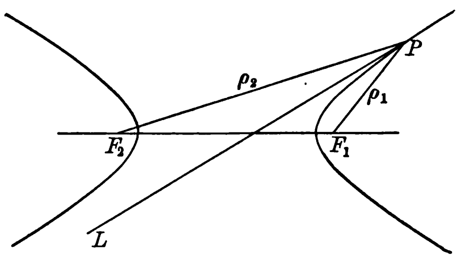
in which ρ_1 and ρ_2 are the focal radii vectores.

For, since (Art. 239) $\rho = e\delta$ and $\delta = x \pm CO = x \pm \frac{a}{e}$, when the origin is at C, the sign $+$ or $-$ being used, according as the focus and directrix at the left or right of C is employed to describe the hyperbola,



$$\therefore \quad \rho = e\left(x \pm \frac{a}{e}\right) \quad \therefore \quad \rho_1 = ex - a, \quad \text{and} \quad \rho_2 = ex + a.$$

305. ***Schol.*** — Subtracting, $\rho_2 - \rho_1 = 2a$, is the equation of the hyperbola in focal coordinates. This equation enables us to construct the hyperbola by continuous motion as follows: in a piece of thread many times longer than $2a$ make a loop so that its knot shall divide the thread into two segments whose difference is $2a$, and fix its extremities at F_1 and F_2, so that $F_1F_2 = 2ae$. Then, if both segments of the thread slide in a notch near the point of a pencil at P, the loop L being held in any direction such as to keep both parts of the



string taut, and paid out as the pencil advances, then the hyperbola is described. For, F_1PL is one segment, and F_2PL the other, and by construction $F_2PL - F_1PL = 2a$, $\therefore$ $F_2P - F_1P = 2a$, or $\rho_2 - \rho_1 = 2a$.

Proposition 7.

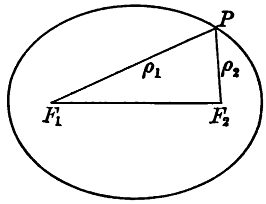
306. ***Theorem.***—*The equations*

$$\rho_1 = a + ex \quad \text{and} \quad \rho_2 = a - ex$$

also represent the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

For, as in Art. 304 $\rho = e\delta$ and $\delta = \frac{a}{e} \pm x$, $\therefore$ $\rho = a \pm ex$.



307. ***Schol.***—Adding, $\rho_1 + \rho_2 = 2a$ is the equation of the ellipse in focal co-ordinates. To construct the ellipse by continuous motion, tie the ends of a thread whose length is $2\left(a + \frac{a}{e}\right)$, and putting it over pins at F_1 and F_2, let the thread run in a notch at the point of a pencil held against it, as at P, the pencil will by its motion describe the ellipse. For, the length of the thread

$= F_1P + F_2P + F_1F_2 = 2a + 2\frac{a}{e}$, and $F_1P + F_2P = 2a$, or $\rho_1 + \rho_2 = 2a$.

Proposition 8.

308. ***Theorem.***—*The equation*

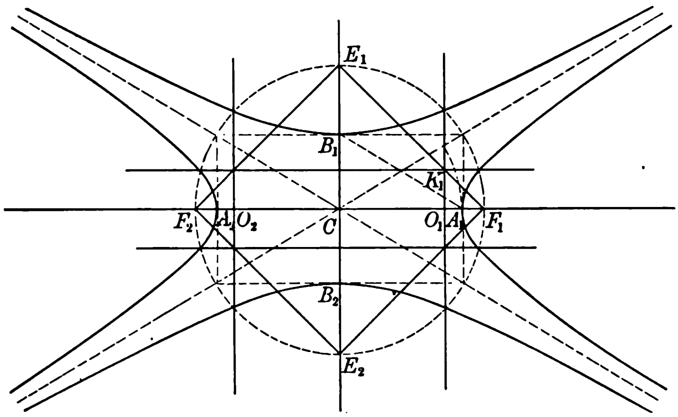
$$x + y = \sqrt{a^2 + b^2}$$

represents a line passing through the focus of an hyperbola and the focus of its conjugate.

Also the equations

$$x = \frac{a^2}{\sqrt{(a^2 + b^2)}} \quad \text{and} \quad y = \frac{b^2}{\sqrt{(a^2 + b^2)}}$$

represent the directrices of an hyperbola and its conjugate.



For (Art. 284), $A_1B_1 = CF_1 = ae_1 = \sqrt{a^2+b^2}$.

Also (Art. 300), $A_1B_1 = CE_1 = be_2 = \sqrt{a^2+b^2}$.

$\therefore$ (Art. 95), $\dfrac{x}{\sqrt{(a^2+b^2)}} + \dfrac{y}{\sqrt{(a^2+b^2)}} = 1$ represents F_1E_1.

Again (Arts. 281, 106), $x = \dfrac{a}{e_1}$ is the directrix of $\dfrac{x^2}{a^2} + \dfrac{y^2}{-b^2} = 1$.

$$\therefore \text{ (Art. 284), } x = \frac{a^2}{\sqrt{(a^2+b^2)}};$$

and similarly, $y = \dfrac{b}{e_2}$ is the directrix of $\dfrac{x^2}{-a^2} + \dfrac{y^2}{b^2} = 1$,

$$\therefore \quad y = \frac{b^2}{\sqrt{(a^2+b^2)}}.$$

309. ***Schol.***—Add these three equations with the signs of the first changed, and they vanish identically; $\therefore$ (Art. 116), the directrices intersect upon the line F_1E_1 at K_1, and the parallelogram of directrices is inscribed in the square $F_1E_1F_2E_2$. These relations enable us to construct the vertices, foci and directrices of an hyperbola and its conjugate; for $ae_1 = be_2 = CF_1 = CE_1 = A_1B_1$,

and $CO_1 : CA_1 :: CA_1 : CF_1$ —*i. e.*, $\dfrac{a}{e_1} : a :: a : ae_1$

give all the relations necessary.

310. ***Example.***—Construct the foci, directrices and vertices of an hyperbola and its conjugate, when $a^2 + b^2 = 13$ and $ab = 6$, a and b being the transverse axes of the hyperbola and its conjugate respectively.

Proposition 9.

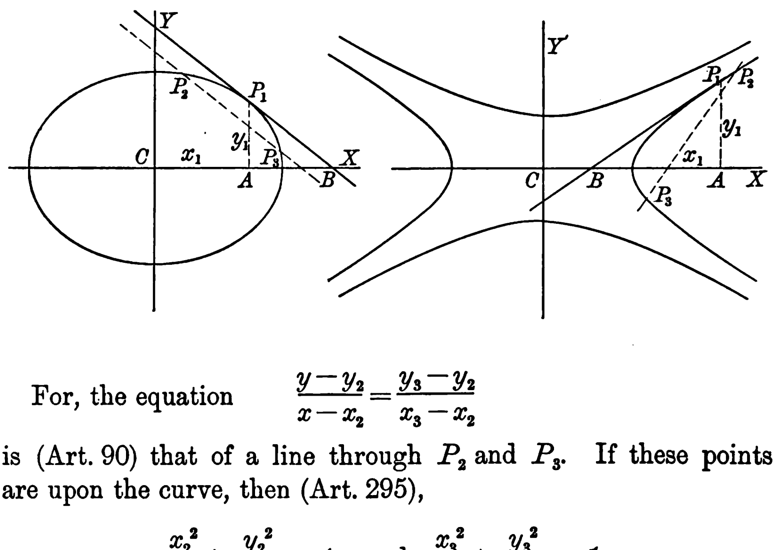
311. ***Theorem.***—*The equation*

$$\frac{xx_1}{a^2} + \frac{yy_1}{\pm b^2} = 1$$

represents a right line tangent to the ellipse or hyperbola

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1;$$

in which x_1 and y_1 are the co-ordinates of the point of tangency, and the origin is at the centre.



For, the equation $\dfrac{y-y_2}{x-x_2} = \dfrac{y_3-y_2}{x_3-x_2}$

is (Art. 90) that of a line through P_2 and P_3. If these points are upon the curve, then (Art. 295),

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{\pm b^2} = 1, \quad \text{and} \quad \frac{x_3^2}{a^2} + \frac{y_3^2}{\pm b^2} = 1,$$

are the equations of condition which subsist. Subtract,

$$\therefore \quad \frac{x_3^2 - x_2^2}{a^2} + \frac{y_3^2 - y_2^2}{\pm b^2} = 0, \quad \therefore \quad \frac{y_3 - y_2}{x_3 - x_2} = -\frac{\pm b^2 (x_3 + x_2)}{a^2 (y_3 + y_2)}.$$

Substituting this in the equation of the line through P_2 and P_3,

$$\therefore \quad \frac{y-y_2}{x-x_2} = -\frac{\pm b^2(x_3+x_2)}{a^2(y_3+y_2)}.$$

Now let $\qquad y_3=y_2=y_1,$ and $x_3=x_2=x_1,$

then
$$\frac{y-y_1}{x-x_1} = -\frac{\pm b^2 x_1}{a^2 y_1};$$

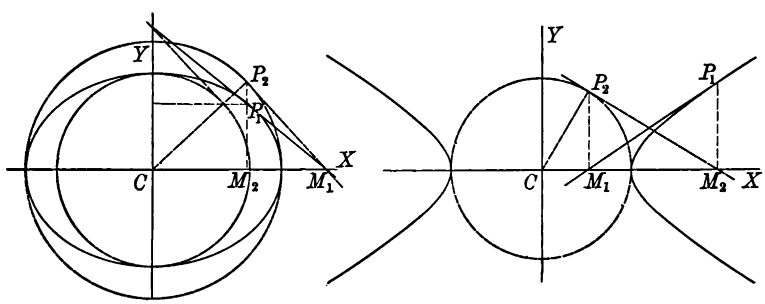
$\therefore$ clearing and transposing, etc.,

$$\frac{xx_1}{a^2}+\frac{yy_1}{\pm b_2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{\pm b^2} \qquad \therefore \quad \frac{xx_1}{a^2}+\frac{yy_1}{\pm b^2}=1.$$

312. ***Cor. 1.***—At the point of contact the equation both of the curve and of its tangent reduces to

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{\pm b^2}=1.$$

313. ***Cor. 2.***—If $y=0$, the intercept of the tangent on x is $BC=x_0=\frac{a^2}{x_1}$. The subtangent in the ellipse is $AB=\frac{a^2}{x_1}-x_1=\frac{a^2-x_1^2}{x_1}$, and in the hyperbola $AB=x_1-\frac{a^2}{x_1}=-\frac{a^2-x_1^2}{x_1}$, $\therefore$ by Art. 184, these subtangents are numerically equal to that of the circle in which $r=a$. This enables us to construct a tangent at (x_1, y_1), as follows.



For the ellipse, draw from P_2 a tangent P_2M_1, and join M_1 and P_1; M_1P_1 is the tangent at P_1. For the hyperbola draw M_2P_2 from M_2 tangent at P_2, and join M_1 and P_1; M_1P_1 is the tangent at P_1.

314. *Example.*—Find the equation of the tangents to the curves $\frac{x^2}{16} + \frac{y^2}{\pm 4} = 1$ at the points whose abscissas are respectively 2 and 6.

$$Ans.\ y = \left(4 - \frac{x}{2}\right) \cot \frac{\pi}{3}, \text{ and } y = \frac{3x - 4}{\sqrt{5}}.$$

315. *Exercise.*—Prove that the equation

$$\frac{xx_1 \pm a(x + x_1)}{a^2} + \frac{yy_1}{\pm b^2} = 0$$

is that of the tangent line when the origin is at the vertex.

Proposition 10.†

316. *Theorem.*—*The equation*

$$y = mx \pm \sqrt{a^2m^2 \pm b^2}$$

also represents a line tangent to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1; \text{ in which } m = \tan\frac{t}{x}.$$

For, the equation $\frac{xx_1}{a^2} + \frac{yy_1}{\pm b^2} = 1$ (Art. 311),

solved with reference to y, is $y = -\frac{\pm b^2 x_1}{a^2 y_1} x + \frac{\pm b^2}{y_1}$ . . . . (*e.*);

$\therefore$ (Art. 104) $\tan\frac{t}{x} = -\frac{\pm b^2 x_1}{a^2 y_1} = m$, and since $\frac{x_1^2}{a^2} + \frac{y_1^2}{\pm b^2} = 1$,

$$\therefore \quad \frac{\pm b^2}{y_1} = \frac{\pm b^2}{y_1}\sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{\pm b^2}},$$

$$\therefore \quad \frac{\pm b^2}{y_1} = \pm\sqrt{a^2\frac{b^4 x_1^2}{a^4 y_1^2} \pm b^2} = \pm\sqrt{a^2m^2 \pm b^2}.$$

Substitute this in (*e.*), $\therefore$ $y = mx + \sqrt{a^2m^2 \pm b^2}$, which is called the magic equation of the tangent line.

317. ***Schol. 1.***—Every equation of this form is tangent to the locus

$$\frac{x^2}{a^2}+\frac{y^2}{\pm b^2}=1.$$

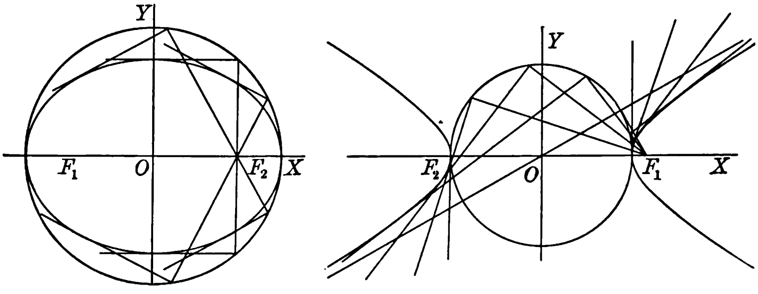
318. ***Schol. 2.***—The equations $\quad y-mx=\sqrt{a^2m^2 \pm b^2}$

$$\text{and}\quad y=-\frac{x}{m}+\sqrt{\frac{a^2}{m^2}\pm b^2},\quad \text{or}\quad my+x=\sqrt{a^2 \pm m^2b^2},$$

represent two tangents perpendicular (Art. 123) to each other. Squaring, adding and dividing by $(1+m^2)$, we obtain $x^2+y^2=a^2 \pm b^2$, which is the equation of the circle which is the locus of the intersection of all tangents perpendicular to each other (cf. Art. 258).

319. ***Schol. 3.***—The equation $y=-\dfrac{1}{m}(x \pm \sqrt{a^2 \mp b^2})$,

or $my+x=\sqrt{a^2 \mp b^2}$, represents a line through the focus (Art. 98) perpendicular to the tangent $y=mx+\sqrt{a^2m^2 \pm b^2}$, the co-ordinates of the focus being $y_1=0$ and $x_1=ae=\sqrt{a^2 \mp b^2}$. Squaring, adding and dividing by $(1+m^2)$, we find $x^2+y^2=a^2$; hence the locus of the foot of the focal perpendicular upon the tangent is a circle whose radius is a.



This principle affords a construction of the ellipse and hyperbola similar to that of the parabola in Art. 257. Draw a circle upon the transverse axis, and through either focus draw chords to the circle; at the extremities of these chords draw perpendiculars to them, and they will be tangent to the ellipse or hyperbola. By drawing a sufficient number the curve may be defined with considerable accuracy.

320. ***Exercises.***—(1.) Prove that the magic equation of the tangent line is $y = m(x \pm a) \pm \sqrt{a^2m^2 + b^2}$ when the origin is at the vertex.

(2.) Find the length of the focal perpendiculars p_1 and p_2 on the tangent, and show that $p_1p_2 = \pm b^2$.

Proposition 11.

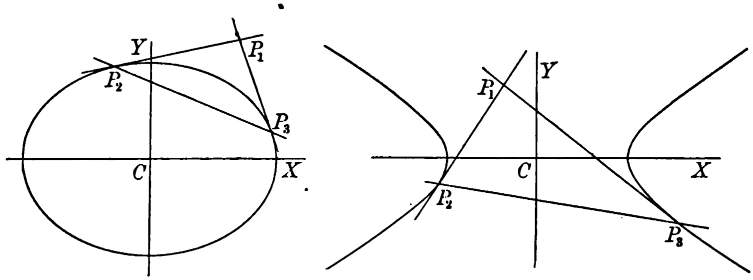
321. ***Theorem.***—*The equation*

$$\frac{xx_1}{a^2} + \frac{yy_1}{\pm b^2} = 1$$

also represents a right line which is the chord of contact of two tangents drawn to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1$$

from the external point (x_1, y_1).



For, let P_2P_3 be the chord of contact of the tangents P_1P_2 and P_1P_3.

From Art. 90, $\dfrac{y - y_3}{x - x_3} = \dfrac{y_2 - y_3}{x_2 - x_3}$ . . . . (*e*.) represents P_2P_3.

From Art. 311, $\dfrac{xx_2}{a^2} + \dfrac{yy_2}{\pm b^2} = 1$ . . . . (*f*.)

represents the tangent at P_2; and if it passes through P_1,

then $$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{\pm b^2}=1 \quad . \; . \; . \; . \quad (g.)$$

Similarly, $$\frac{x_1x_3}{a^2}+\frac{y_1y_3}{\pm b^2}=1 \quad . \; . \; . \; . \quad (h.)$$

Subtract (*h.*) from (*g.*), $\therefore \quad \dfrac{x_1(x_2-x_3)}{a^2}+\dfrac{y_1(y_2-y_3)}{\pm b^2}=0.$

$$\therefore \quad \frac{y_2-y_3}{x_2-x_3}=\frac{\mp b^2x_1}{a^2y_1}, \quad \therefore \quad \text{from } (e). \quad \frac{y-y_3}{x-x_3}=\frac{\mp b^2x_1}{a^2y_1}$$

$$\therefore \quad \frac{xx_1}{a^2}+\frac{yy_1}{\pm b^2}=\frac{x_1x_3}{a^2}+\frac{y_1y_3}{\pm b^2}, \quad \therefore \quad \text{from } (h.) \quad \frac{xx_1}{a^2}+\frac{yy_1}{\pm b^2}=1$$

represents P_2P_3.

P_1 is called the **pole**, and P_2P its **polar**, with respect to the ellipse or hyperbola.

322. ***Examples.***—(1.) The semi-axes of an ellipse are *4* and *2*. Find the intercepts of the polar of the point $(-5, 6)$.

Ans. $x_0=-3\frac{1}{5}$, $y_0=\frac{2}{3}$.

(2.) Apply the data of the last example to an hyperbola by using $2\sqrt{-1}$ instead of *2* for the semi-conjugate axis.

Ans. $x_0=-3\frac{1}{5}$, $y_0=-\frac{2}{3}$.

323. ***Exercise.***—When the pole is on the directrix, the polar is a focal chord.

Proposition 12.

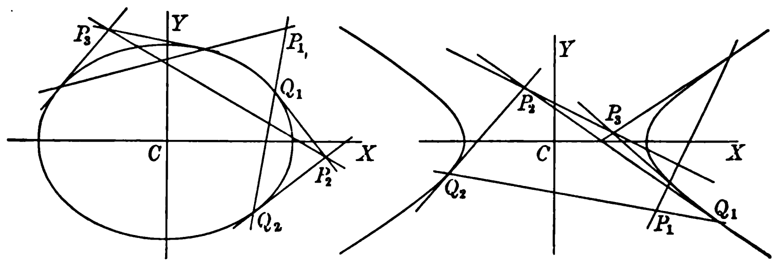
324. ***Theorem.***—*The equation*

$$\frac{xx_1}{a^2}+\frac{yy_1}{\pm b^2}=1$$

also represents the locus of the intersections of the pairs of tangents to the conics

$$\frac{x^2}{a^2}+\frac{y^2}{\pm b^2}=1,$$

drawn from the extremities of all chords which pass through any fixed point (x_1, y_1).



For, let P_1 be the fixed point through which all the chords pass, and let Q_1Q_2 be any chord through P_1. If the tangents at Q_1 and Q_2 meet in some point P_2,

then $$\frac{xx_2}{a^2} + \frac{yy_2}{\pm b^2} = 1$$

is the equation of the chord Q_1Q_2 by Art. 321, and at P_1 this equation becomes

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{\pm b^2} = 1 \quad \ldots . \quad (k.)$$

Similarly, if the tangents at the extremity of another chord through P_1 intersect at P_3,

$$\frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} = 1 \quad \ldots . \quad (l.)$$

But (Art. 90) $$\frac{y - y_3}{x - x_3} = \frac{y_2 - y_3}{x_2 - x_3} \quad \ldots (i.) \text{ represents } P_2P_3.$$

Eliminate as in Art. 321, $\therefore \ \dfrac{xx_1}{a^2} + \dfrac{yy_1}{\pm b^2} = 1$ represents P_2P_3, which is the locus of the intersections of all pairs of tangents at the extremities of chords through P_1.

P_1 is called the **pole**, and P_2P_3 its **polar**, with respect to the ellipse or hyperbola.

325. *Examples.*—Find the polars of the following points, with reference to the ellipse $x^2+4y^2=16$, and the hyperbola $x^2-4y^2=16$.

(1.) $(2, 1.)$ *Ans.* $y=-\frac{1}{2}x+4$, and $y=\frac{1}{2}x-4$.

(2.) $(5, -7.)$ *Ans.* $y=\frac{5x}{28}-\frac{4}{7}$, and $y=-\frac{5x}{28}+\frac{4}{7}$.

(3.) $(2, 0.)$ *Ans.* $x=8$.

(4.) $(0, 0.)$

326. *Exercise.*—If the focus be the pole, the directrix will be the polar.

Proposition 13.

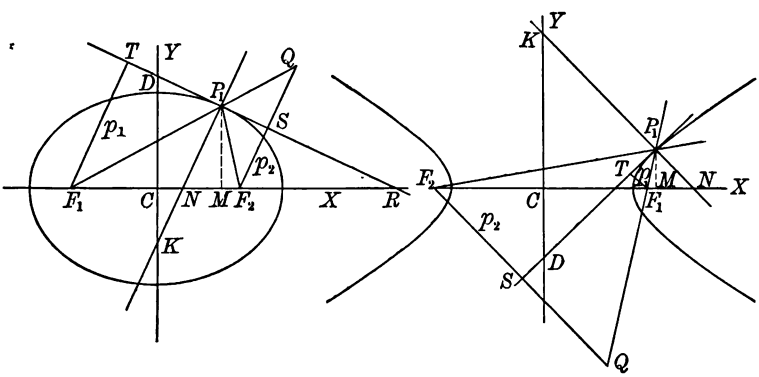
327. *Theorem.*—*The equation*

$$\frac{y-y_1}{x-x_1}=\frac{a^2y_1}{\pm b^2x_1}$$

represents a normal line to the conic

$$\frac{x^2}{a^2}+\frac{y^2}{\pm b^2}=1$$

at the point (x_1, y_1).



For, since (Art. 311) $\frac{y-y_1}{x-x_1}=-\frac{\pm b^2x_1}{a^2y_1}$ is one form of the

equation of the tangent line, $\therefore$ (Art. 128), $\dfrac{y-y_1}{x-x_1}=\dfrac{a^2y_1}{\pm b^2x_1}$ is the equation of a line perpendicular to the tangent at the point $(x_1,\ y_1)$ — that is, the normal.

328. ***Cor.***—If $y=0$, the intercept of the normal is

$$x=x_1\left(1-\frac{\pm b^2}{a^2}\right)=x_1e^2=CN,$$

and the *subnormal* $\quad =x_1-x_1\left(1-\dfrac{\pm b^2}{a^2}\right)=\dfrac{\pm b^2}{a^2}x_1=NM.$

Similarly, if $x=0$, the intercept is $y=y_1\left(1-\dfrac{a^2}{\pm b^2}\right)=CK.$

329. ***Schol. 1.***—In the hyperbola, by Arts. 283 and 328,

$$F_1N=F_1C+CN=-ae+e^2x_1,$$

and (Art. 9) $\quad F_2N=F_2C+CN=ae+e^2x_1.$

But (Art. 304), $F_1P_1=-a+ex_1$, and $F_2P_1=a+ex_1$.

$$\therefore\ \frac{F_1N}{F_1P_1}=\frac{F_2N}{F_2P_1}=e.\quad \therefore\ F_1N:F_2N::F_1P_1:F_2P_1.$$

In the ellipse, by Arts. 283 and 328,

$$F_1N=F_1C+CN=ae+e^2x_1,$$

$$NF_2=NC+CF_2=ae-e^2x_1,$$

$$\therefore\ \frac{F_1N}{F_1P_1}=\frac{NF_2}{F_2P_1}=e,\quad \therefore\ F_1N:NF_2::F_1P_1:F_2P_1.$$

$\therefore$ By geometry, NP_1, the normal of an ellipse, bisects $F_1P_1F_2$, the internal angle of the focal radii of P_1.

$$\therefore\ F_1P_1N=NP_1F_2.\quad \text{Also, } DP_1N=NP_1R.$$

Subtract, $\quad \therefore\ F_2P_1R=DP_1F_1=RP_1Q,$

$\therefore$ the tangent bisects FP_1Q the external angle of the focal radii. Similarly, the normal of the hyperbola bisects the external and the tangent the internal angle of the focal radii.

Conversely, a tangent being drawn to an ellipse or hyperbola, to find the point of contact. Through F_2 draw F_2Q perpendicular to the tangent, and make $QS = SF_2$: through Q draw QF_1, and the point P_1, in which it cuts the curve and tangent, will be the point of contact.

330. ***Schol. 2.*** †—Solve the equation of the normal for y,

$$\therefore \quad y = \frac{a^2 y_1}{\pm b^2 x_1} x - (a^2 \pm b^2) \frac{y_1}{\pm b^2}.$$

Let
$$\tan \frac{n}{x} = m = \frac{a^2 y_1}{\pm b^2 x_1},$$

then
$$\frac{y_1}{\pm b^2} = \frac{\frac{y_1}{\pm b^2}}{\sqrt{\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{\pm b^2}\right)}} = \frac{\frac{a^2 y_1}{\pm b^2 x_1}}{\frac{a^2}{x_1}\sqrt{\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{\pm b^2}\right)}}$$

$$= \frac{\frac{a^2 y_1}{\pm b^2 x_1}}{\sqrt{\left(a^2 \pm b^2 \frac{a^4 y_1^2}{b^4 x_1^2}\right)}} = \frac{m}{\sqrt{(a^2 \pm b^2 m^2)}},$$

$$\therefore \quad y = mx - \frac{m(a^2 \pm b^2)}{\sqrt{(a^2 \pm b^2 m^2)}}$$

is the *magic equation* of the normal.

331. ***Exercises.***—(1.) Show that the equation $\frac{y - y_1}{x - x_1} = \frac{a^2 y_1}{\pm b^2 x_1}$ also represents a line through the pole (x_1, y_1) perpendicular to the polar $\frac{x x_1}{a^2} + \frac{y y_1}{\pm b^2} = 1$, with respect to the curve $\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1$ (Art. 128).

(2.) Show that in the ellipse, $P_1N = \frac{b}{a}\sqrt{a^2 - e^2 x_1^2}$,

and that
$$P_1K = \frac{a}{b}\sqrt{a^2 - e^2 x_1^2},$$

$$\therefore \quad P_1N \,.\, P_1K = a^2 - e^2 x_1^2,$$

$$\therefore \quad \text{(Art. 306)} \qquad \rho_1 \rho_2 = P_1N \,.\, P_1K.$$

(3.) Show that in the hyperbola also $\rho_1\rho_2 = P_1N \cdot P_1K$.

Proposition 14.

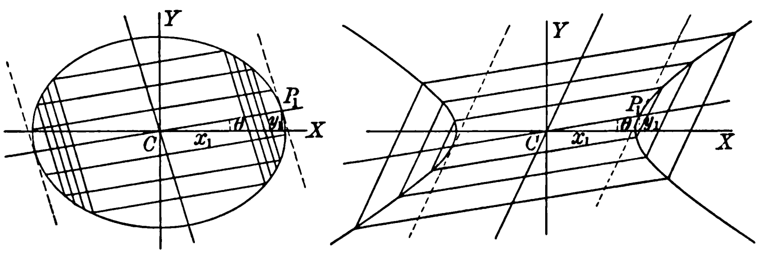
332. ***Theorem.***—*The equation*

$$\frac{x \cos\theta}{a^2} + \frac{y \sin\theta}{\pm b^2} = 0$$

represents that diameter of the curves

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1$$

which bisects the system of parallel chords each of which makes the angle θ *with the axis of* x.



For, from Art. 100, if $\dfrac{y - y_1}{\sin\theta} = \dfrac{x - x_1}{\cos\theta} = l$, then l is the distance from any point (x_1, y_1) upon the line $y - b_1 = x \tan\theta$, to some point (x, y) at the intersection of this line with

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1; \qquad \therefore \quad x = x_1 + l\cos\theta, \text{ and } y = y_1 + l\sin\theta.$$

Substitute these values of x and y, in (*bc.*).

$$\therefore \quad \frac{(x_1 + l\cos\theta)^2}{a^2} + \frac{(y_1 + l\sin\theta)^2}{\pm b^2} = 1.$$

Expand,

$$\therefore \quad l^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{\pm b^2}\right) + 2l\left(\frac{x_1\cos\theta}{a^2} + \frac{y_1\sin\theta}{\pm b^2}\right) + \frac{x_1^2}{a^2} + \frac{y_1^2}{\pm b^2} = 1.$$

There are evidently two values of l. Let these values be equal numerically, but of opposite signs; then the point (x_1, y_1) must bisect the chord, and, by the "General Theory of Equations," the coefficient of the second term—*i. e.*, the coefficient of the first power of l — must vanish; for the quadratic is the product of the sum and difference of the same quantities.

$$\therefore \quad \frac{x_1 \cos \theta}{a^2} + \frac{y_1 \sin \theta}{\pm b^2} = 0,$$

in which (x_1, y_1) is the middle point only of any chord. Now, making (x_1, y_1) general,

$$\therefore \quad \frac{x \cos \theta}{a^2} + \frac{y \sin \theta}{\pm b^2} = 0$$

is the locus of the middle points of all these parallel chords (Art. 242), and is evidently a right line through the origin—that is, through the centre.

333. ***Schol.***—When $x_1 = 0$, and $y_1 = 0$, we have the distance from the origin to the curve $\frac{1}{l} = \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{\pm b^2}\right)^{\frac{1}{2}}$, $\therefore$ l is always real in the ellipse, and is real in the hyperbola when $\frac{\cos^2 \theta}{a^2} > \frac{\sin^2 \theta}{b^2}$—*i. e.*, when $\tan \theta < \pm \frac{b}{a}$, but imaginary when $\tan \theta > \pm \frac{b}{a}$—that is, in the latter case the line in the given direction does not cut the curve

$$\frac{x^2}{a^2} + \frac{y^2}{-b^2} = 1, \text{ but does cut } \frac{x^2}{-a^2} + \frac{y^2}{b^2} = 1.$$

334. ***Examples.***—(1.) The semi-axes of an ellipse are 4 and 3; find the equation of the bisectrix of a system of chords making an angle $\tan^{-1}\sqrt{3}$ with the axis of x.

$$\textit{Ans. } y = -\frac{9}{16\sqrt{3}}x.$$

(2.) Find the same for the hyperbola whose semi-axes are 4 and $3\sqrt{-1}$.

$$\textit{Ans. } y = \frac{3\sqrt{3}}{16}x.$$

Proposition 15.

335. ***Theorem.***—*The equation*

$$m_1 m_2 = -\frac{\pm b^2}{a^2} \text{ expresses}$$

1st. The condition that the diameter $y = m_2 x$ *shall bisect the system of chords* $y - b_1 = m_1 x$, *(in which* b_1 *is a variable constant).*

2d. The condition that $y = m_1 x$ *and* $x = m_2 x$ *shall be conjugate diameters.*

3d. The fact that $y - y_1 = m_2 (x - x_1)$, *the tangent at the extremity* (x_1, y_1) *of the diameter* $y = m_1 x$, *is parallel to its conjugate diameter* $y = m_2 x$, *and to the system of chords* $y = m_1 x + b_2$.

4th. The fact that supplemental chords $y - y_1 = m_1 (x - x_1)$ *and* $y + y_1 = m_2 (x + x_1)$ *are always parallel to conjugate diameters* $y = m_1 x$ *and* $y = m_2 x$,

1st. The equation of the diameter (Art. 332),

$$\frac{x \cos \theta}{a^2} + \frac{y \sin \theta}{\pm b^2} = 0,$$

solved for y, is $y = -\frac{\pm b^2}{a^2} x \cot \theta$, or $y = m_2 x$. The equation of the chords it bisects is $y = x \tan \theta + b_1$, or $y = m_1 x + b_1$,

$$\therefore \quad m_1 m_2 = -\frac{\pm b^2}{a^2}.$$

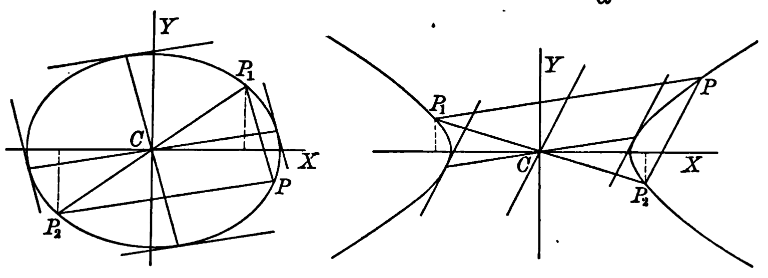
2d. When $y = x \tan \theta + b_1$, or $y = m_1 x + b_1$ is the system of chords, then (by 1st), $y = -x \cdot \frac{\pm b^2}{a^2 \tan \theta}$, or $y = m_2 x$ is their diameter—*i. e.*, the coefficient of x for the diameter is obtained from the coefficient of x for the chord by multiplying its reciprocal by $-\frac{\pm b^2}{a^2}$.

Similarly, when $\quad y = -x \cdot \dfrac{\pm b^2}{a^2 \tan\theta} + b_2$, or $y = m_2 x + b_2$,
is the system of chords; multiply the reciprocal of the coefficient of x by $-\dfrac{\pm b^2}{a^2}$, $\therefore$ $y = x \tan\theta$, or $y = m_1 x$ is the diameter, which is parallel (Art. 128) to the chords $y = m_1 x + b_1$,

$\therefore$ (Art. 242) $y = m_1 x$ and $y = m_2 x$ are conjugate,

but
$$m_1 m_2 = -\frac{\pm b^2}{a^2}.$$

3d. Now, $y - y_1 = -\dfrac{\pm b^2 x_1}{a^2 y_1}(x - x_1)$, or $y - y_1 = m_2(x - x_1)$ represents (Art. 317) the tangent at (x_1, y_1), and $y = \dfrac{y_1}{x_1}x$, or $y = m_1 x$, is the diameter through (x_1, y_1), $\therefore$ $m_1 m_2 = -\dfrac{\pm b^2}{a^2}$.



4th. When P is *any* point, $y - y_1 = m_1(x - x_1)$ represents PP_1.
Also (since $y_2 = -y_1$), $y + y_1 = m_2(x + x_1)$ represents PP_2.
Multiply, $\quad \therefore \quad y^2 - y_1^2 = m_1 m_2 (x^2 - x_1^2) \quad . \; . \; . \quad (m.)$

Again, if P_1 and P_2 are on the curve, then $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{\pm b^2} = 1$,

and if P or (x, y) is also on the curve, $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{\pm b^2} = 1$.

Subtract, $\therefore$ $\dfrac{x^2 - x_1^2}{a^2} + \dfrac{y^2 - y_1^2}{\pm b^2} = 0$, or $y^2 - y_1^2 = -\dfrac{\pm b^2}{a^2}(x^2 - x_1^2)$,
which (Art. 112) is the equation of the two chords PP_1 and PP_2 respectively. Divide the equation by equation $(m.)$,

$$\therefore \quad m_1 m_2 = -\frac{\pm b^2}{a^2}.$$

336. *Schol.*—The conjugate diameters in the ellipse both cut the curve (Art. 333), and since $m_1m_2 = -\frac{b^2}{a^2}$, if m_1 is the tangent of an acute angle it is $+$, and m_2 is $-$, and is then the tangent of an obtuse angle, and vice versa. In the circle $a = b$, $\therefore$ $m_1m_2 = -1$ (cf. Art. 123).

The angle between the axis of x and the conjugate diameters in the hyperbola are both acute, or both obtuse, since $m_1m_2 = +\frac{b^2}{a^2}$, and if $m_1 > \pm\frac{b}{a}$, $\therefore$ $m_2 < \pm\frac{b}{a}$. Hence (Art. 333), only one of the conjugates intersects the hyperbola.

337. *Examples.*—(1.) Given an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and a diameter making an angle of $30°$ with the axis of x; to find its conjugate.

Ans. $16y = -9\sqrt{3}.x$.

(2.) Find the equation of a tangent to the same ellipse parallel to the line $\frac{x}{3.5} - \frac{y}{7} = 1$. *Ans.* $y = 2x \pm 8.544$.

(3.) Perform the same operations with the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

338. *Exercise.*—Construct conjugate diameters; also a tangent parallel to a given line.

Proposition 16.

339. *Theorem.*—*The equation*

$$\frac{x^2}{a_1^2} + \frac{y^2}{\pm b_1^2} = 1$$

represents an ellipse or hyperbola referred to conjugate diameters as co-ordinate axes; in which a_1^2 *and* $\pm b_1^2$ *are the squares of the semi-conjugate diameters.*

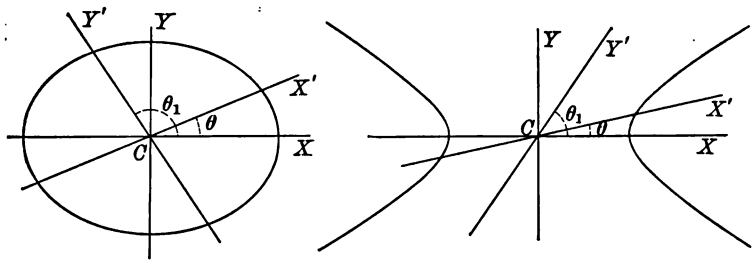
For, change the direction of the axes in the curve

$$\frac{x^2}{a^2} + \frac{y^2}{\pm b^2} = 1.$$

By Art. 71, $x = x' \cos\theta + y' \cos\theta_1$, and $y = x' \sin\theta + y' \sin\theta_1$.

$$\therefore \quad \frac{(x' \cos\theta + y' \cos\theta_1)^2}{a^2} + \frac{(x' \sin\theta + y' \sin\theta_1)^2}{\pm b^2} = 1,$$

or
$$\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{\pm b^2}\right)x'^2 + \left(\frac{\cos^2\theta_1}{a^2} + \frac{\sin^2\theta_1}{\pm b^2}\right)y'^2$$
$$+ 2\left(\frac{\cos\theta\cos\theta_1}{a^2} + \frac{\sin\theta\sin\theta_1}{\pm b}\right)x'y' = 1.$$



But since these are conjugate diameters, by Art. 335,

$$\tan\theta \tan\theta_1 + \frac{\pm b^2}{a^2} = 0, \quad \therefore \text{ by trig. } \frac{\cos\theta\cos\theta_1}{a^2} + \frac{\sin\theta\sin\theta_1}{\pm b^2} = 0.$$

$$\therefore \quad \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{\pm b^2}\right)x'^2 + \left(\frac{\cos^2\theta_1}{a^2} + \frac{\sin^2\theta_1}{\pm b^2}\right)y'^2 = 1.$$

But (Art. 333), $\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{\pm b^2} = \dfrac{1}{a_1^{\,2}}$, and $\dfrac{\cos^2\theta_1}{a^2} + \dfrac{\sin^2\theta_1}{\pm b^2} = \dfrac{1}{\pm b_1^{\,2}}$;

$$\therefore \quad \frac{x'^2}{a_1^{\,2}} + \frac{y'^2}{\pm b_1^{\,2}} = 1.$$

340. ***Exercise.***—Prove that $\dfrac{xx_1}{a_1^{\,2}} + \dfrac{yy_1}{\pm b_1^{\,2}} = 1$ is the equation of the tangent referred to conjugate diameters.

Proposition 17.

341. ***Theorem****—The equations*

$$\frac{x_1}{a} = \pm\frac{y_2}{b}, \quad \text{and} \quad \frac{x_2}{a} = \pm\frac{y_1}{b}$$

express the relations between the co-ordinates of the extremities (x_1, y_1) *and* (x_2, y_2) *of conjugate diameters.*

For, if $\frac{x}{x_1} = \frac{y}{y_1}$ is the equation of a diameter whose extremity is (x_1, y_1), by Art. 335, 2d, its conjugate is $\frac{xx_1}{a^2} + \frac{yy_1}{\pm b^2} = 0$.

$\therefore$ at (x_2, y_2), we have $\frac{x_1x_2}{a^2} + \frac{y_1y_2}{\pm b^2} = 0$, and $\frac{x_2^2}{a^2} + \frac{y_2^2}{\pm b^2} = 1$.

Eliminate y_2, $\therefore \frac{x_2^2}{a^2} + \frac{b^4x_1^2x_2^2}{\pm b^2a^4y_1^2} = 1$, or $\frac{x_2^2}{a^2} \cdot \frac{b_2^2}{y_1^2}\left(\frac{y_1^2}{\pm b^2} + \frac{x_1^2}{a^2}\right) = 1$;

$$\therefore \frac{x_2^2}{a^2} \cdot \frac{b^2}{y_1^2} = 1, \qquad \therefore \frac{x_2}{a} = \pm \frac{y_1}{b}.$$

Similarly, it may be shown that $\frac{x_1}{a} = \pm \frac{y_2}{b}$.

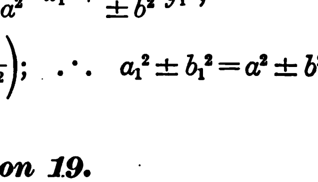
Proposition 18.

342. *Theorem.—The equation*

$$a_1^2 \pm b_1^2 = a^2 \pm b^2$$

is an equation of condition of conjugate diameters.

For, $a_1^2 = x_1^2 + y_1^2$, and $b_1^2 = x_2^2 + y_2^2$; and taking the values of x_2 and y_2 from Art. 341, $b_1^2 = \frac{b^2}{a^2} x_1^2 + \frac{a^2}{b^2} y_1^2$.



$$\therefore a_1^2 \pm b_1^2 = x_1^2 + y_1^2 + \frac{\pm b^2}{a^2} x_1^2 + \frac{a^2}{\pm b^2} y_1^2;$$

$$\therefore a_1^2 \pm b_1^2 = (a^2 \pm b^2)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{\pm b^2}\right); \quad \therefore a_1^2 \pm b_1^2 = a^2 \pm b^2.$$

Proposition 19.

343. *Theorem.—The equation*

$$\pm b_1^2 = \rho_1\rho_2$$

expresses the square of the length of a semi-diameter conjugate to a_1, when ρ_1 and ρ_2 are the focal radii of the extremity of a_1.

For, $\pm b_1^2 = a^2 \pm b^2 - a_1^2$, by Art. 342,

$$\therefore \quad \pm b_1^2 = a^2 \pm b^2 - (x_1^2 + y_1^2) = a^2 \pm b^2 - x_1^2 - \frac{\pm b^2}{a^2}(a^2 - x_1^2),$$

$$\therefore \quad \pm b_1^2 = a^2 - \left(1 - \frac{\pm b^2}{a^2}\right) x_1^2 = a^2 - e^2 x_1^2.$$

But (Arts. 304, 306) $\rho_1 \rho_2 = \pm (a^2 - e^2 x_1^2)$. $\therefore \quad \rho_1 \rho_2 = \pm b_1^2$.

Proposition 20.†

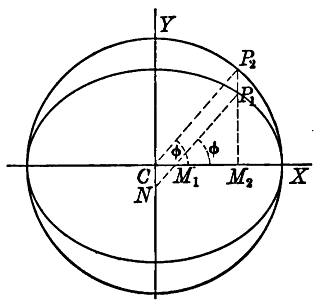
344. *The equations*

$$\frac{x}{a} = \cos \varphi \quad \text{and} \quad \frac{y}{b} = \sin \varphi$$

together represent an ellipse; in which φ ***is the eccentric angle.***

For, squaring and adding the equations, we have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \sin^2 \varphi + \cos^2 \varphi = 1$, by trig. This equation represents an ellipse, hence $\dfrac{x}{a} = \cos \varphi$, and $\dfrac{y}{b} = \sin \varphi$, must together represent the same.

345. ***Schol. 1.***—The eccentric angle of P_1 is XCP_2; for since the ellipse is a projection of the circle, by Art. 297, $\dfrac{y_c}{y_e} = \dfrac{a}{b}$, and by sim. tri's $\dfrac{y_c}{y_e} = \dfrac{CP_2}{M_1P_1} = \dfrac{a}{M_1P_1}$.



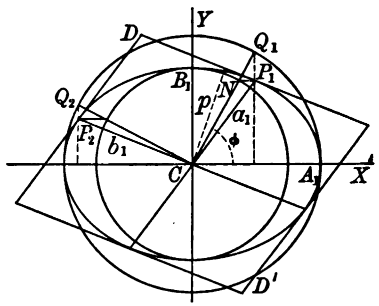
$$\therefore \quad \frac{y_c}{y_e} = \frac{a}{b} = \frac{a}{M_1P_1},$$

$$\therefore \quad b = M_1P_1, \text{ and } \frac{M_2P_1}{M_1P_1} = \frac{y}{b} = \sin \varphi;$$

similarly, if $CP_2 = a$, then $\dfrac{x}{a} = \cos \varphi$.

346. ***Schol. 2.***—Since in the parallelogram P_1P_2CN we have $CP_2=NP_1=a$, and $M_1P_1=b$, $\therefore$ $NM_1=a-b$.

This principle is employed in the construction of the trammel or elliptic compasses. For, if M_1 and N run upon the axes as guides, any point P_1 of the line M_1N will describe an ellipse. The point halfway between M and N describes a circle, a particular case of the ellipse.



347. ***Schol. 3.***—The point P_1 corresponding to the eccentric angle φ is most easily constructed as in the figure. This affords a good method of constructing an ellipse by points; for since $y_c=a\sin\varphi$, and $y_e=b\sin\varphi$, we have by subtraction

$$y_c-y_e=(a-b)\sin\varphi.$$

Hence, draw two concentric circles whose radii are respectively a and b, and from the points Q_1 and N where any radius cuts the two circles, draw parallels to the axes of y and x respectively as represented; their intersection will be a point of an ellipse whose semi-axes are a and b.

348. ***Exercise.***—Show that $\frac{x}{a}\cos\varphi_1+\frac{y}{b}\sin\varphi_1=1$ is the equation of a line tangent to the ellipse at a point whose eccentric angle is φ_1.

Proposition 21.†

349. ***Theorem.***—*The equations*

$$\frac{x}{a}=\sec\varphi \quad \text{and} \quad \frac{y}{b}=\tan\varphi$$

together represent an hyperbola; in which φ is the eccentric angle.

For, by squaring and subtracting $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\sec^2\varphi-\tan^2\varphi=1$, by trig., which represents an hyperbola;

hence $\quad \frac{x}{a}=\sec\varphi$, and $\frac{y}{b}=\tan\varphi$, must represent the same.

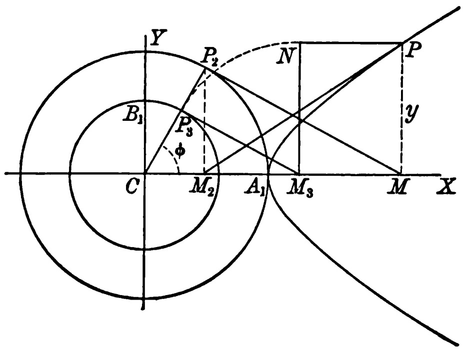
350. ***Schol. 1.***—The eccentric angle of P is XCP_2. For, if $CP_2 = a$, then $CM = x = a \sec \varphi$; and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes $\sec^2 \varphi - \frac{y^2}{b^2} = 1$, or $\frac{y}{b} = \sqrt{\sec^2 \varphi - 1} = \tan \varphi$, by trig.

Also, if $CP_3 = b$,

then $\frac{P_3M_3}{b} = \tan \varphi$,

$\therefore\ XCP_2 = \varphi$:

$\therefore$ also $P_3M_3 = y$.



351. ***Schol. 2.***—The eccentric angle affords a method for constructing the hyperbola by points.

Draw two concentric circles with the radii a and b, and from the points where any radius cuts the two circles draw the tangents P_2M, P_3M_3. Then $M_3N = M_3P_3 = y$ and $CM = x$. Through M and N draw parallels to the axes y and x respectively; their intersection at P is a point of the hyperbola.

352. ***Exercise.***—Show that the line

$$\frac{x}{a} \sec \varphi_1 - \frac{y}{b} \tan \varphi_1 = 1$$

is a tangent to the hyperbola at the point whose eccentric angle is φ_1.

353. ***Schol.***—The eccentric angle of any point P of the conic $\frac{x^2}{a^2} = \frac{y^2}{\pm b^2} = 1$ is included between the axis of x and that radius of the circle $x^2 + y^2 = a^2$ which has the same subtangent as P (Art. 313).

Proposition 22.†

354. ***Theorem.***—*The equation*

$$\varphi_1 \mp \varphi_2 = \frac{\pi}{2}$$

is also the equation of condition for conjugate diameters of the ellipse and hyperbola; in which φ_1 and φ_2 are the eccentric angles of the extremities of the conjugates.

1st. In the ellipse (Art. 344), $\frac{x}{a}=\cos\varphi$ and $\frac{y}{b}=\sin\varphi$,

and (Art. 335), $$m_1m_2=-\frac{b^2}{a^2}.$$

But by trig., $$m_1=\frac{y_1}{x_1}=\frac{b\sin\varphi_1}{a\cos\varphi_1}=\frac{b}{a}\tan\varphi_1,$$

and $$m_2=\frac{y_2}{x_2}=\frac{b\sin\varphi_2}{a\cos\varphi_2}=\frac{b}{a}\tan\varphi_2,$$

$$\therefore\quad m_1m_2=-\frac{b^2}{a^2}=\frac{b^2}{a^2}\tan\varphi_1\tan\varphi_2.$$

$$\therefore\quad \tan\varphi_1\tan\varphi_2=-1.\quad\therefore\quad \text{(Art. 123)},\qquad \varphi_1-\varphi_2=\frac{\pi}{2}.$$

2d. In the hyperbola $\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=1$, $\frac{x}{a}=\sec\varphi$, and $\frac{y}{b}=\tan\varphi$.

Similarly in the conjugate hyperbola $\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2=1$,

$$\frac{y}{b}=\operatorname{cosec}\varphi,\quad\text{and}\quad\frac{x}{a}=\cot\varphi.$$

Also $$m_1=\frac{y_1}{x_1}=\frac{b\tan\varphi_1}{a\sec\varphi_1}=\frac{b}{a}\sin\varphi_1,$$

and $$m_2=\frac{y_2}{x_2}=\frac{b\operatorname{cosec}\varphi_2}{a\cot\varphi_2}=\frac{b}{a}\sec\varphi_2.$$

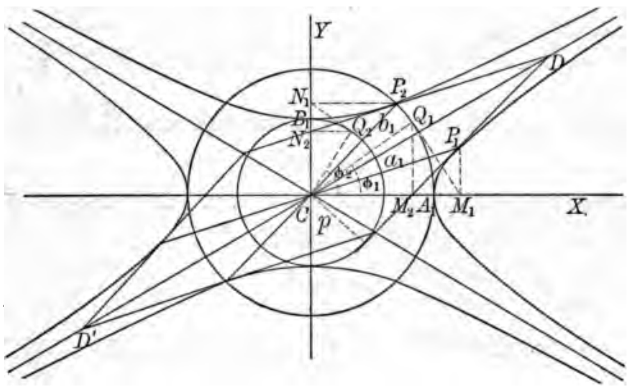
For, if $y=m_1x$ cut the primary hyperbola, then $y=m_2x$ cuts the conjugate, by Arts. 333 and 337, when $m_1m_2=\frac{b^2}{a^2}$.

$$\therefore\quad m_1m_2=\frac{b^2}{a^2}=\frac{b^2}{a^2}\sin\varphi_1\sec\varphi_2;$$

$$\therefore\quad \sin\varphi_1\sec\varphi_2=\frac{\sin\varphi_1}{\cos\varphi_2}=1;\quad\therefore\quad\text{by trig.}\quad \varphi_1+\varphi_2=\frac{\pi}{2}.$$

355. *Schol.*—This enables us (see fig. of Art. 347) to construct conjugate diameters in an ellipse by drawing two radii of the circle $r=a$ at right angles. The ordinates of their extremities Q_1 and Q_2 will give the extremities P_1 and P_2 of the conjugate diameters.

Conjugate diameters in the hyperbola may be constructed as follows:



Draw two circles with a and b for radii, and let the angle $XCY = XCQ_1 + XCQ_2 = \frac{\pi}{2}$. The points P_1 and P_2, having the same subtangents M_1M_2 and N_1N_2 as the arcs A_1Q_1 and B_1Q_2, will be the extremities of the conjugate diameters.

Proposition 23.

356. *Theorem.—The equation*

$$a_1 b_1 \sin\beta = ab$$

expresses the fact that the tangents through the extremities of any pair of conjugate diameters enclose the same area; in which $\beta = \theta_1 - \theta_2$ is the angle between the semi-conjugate axes a_1 and b_1.

Let $\frac{xx_1}{a^2} + \frac{yy_1}{\pm b^2} = 1$ be the tangent line through (x_1, y_1), the extremity of the diameter $y = m_1 x$, and parallel to the diameter

$$\frac{xx_1}{a^2} + \frac{yy_1}{\pm b^2} = 0.$$

By Art. 142, the length of the perpendicular from the origin on the tangent is
$$p=\frac{1}{\sqrt{\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)}}.$$

But (Arts. 341, 342),
$$b_1=\sqrt{x_2^2+y_2^2}=\sqrt{\frac{b^2}{a^2}x_1^2+\frac{a^2}{b^2}y_1^2}=ab\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}.$$

Now, in the figs. of Arts. 347 and 355 the area of the parallelogram $CP_1DP_2=pb_1$, but $pb_1=ab$.

Again, by trig., $CP_1DP_2=a_1b_1\sin\beta\quad\therefore\quad ab=a_1b_1\sin\beta$.

Now $4ab$ = rectangle of the axes.

And $4a_1b_1\sin\beta$ = rectangle DD'.

357. ***Schol.***—By Art. 356, $\sin^2\beta=\dfrac{a^2b^2}{a_1^2b_1^2}=\dfrac{4a^2b^2}{(a_1^2+b_1^2)^2-(a_1^2-b_1^2)^2}$;

$$\therefore\quad \text{(Art. 342)}\ \sin^2\beta=\frac{4a^2b^2}{(a^2+b^2)^2-(a_1^2-b_1^2)^2}\ \text{for the ellipse}\ldots\ (\textit{n.})$$

and
$$\sin^2\beta=\frac{4a^2b^2}{(a_1^2+b_1^2)^2-(a^2-b^2)^2}\ \text{for the hyperbola}\ldots.\ (\textit{o.})$$

In the ellipse, when $a_1=b_1$, $\sin\beta=\dfrac{\pm 2ab}{a^2+b^2}$ is a minimum value of (*n.*).

These are the **equi-conjugate diameters**; and since the ellipse is symmetrical, $m_1=-m_2$ in the equation $m_1m_2=-\dfrac{b^2}{a^2}$,

and $\therefore\ m=\pm\dfrac{b}{a}$ for the equi-conjugate diameters. $\therefore$ they lie in the diagonals of the rectangle of the axes.

In the hyperbola, $\sin\beta=0$ is a minimum value, for a_1 increases with b_1, and the denominator of (*o.*) can be made infinite. These conjugates are infinite, and coincide. Since $m_1m_2=\dfrac{b^2}{a^2}$,

when $m_1=m_2$, then $m=\pm\dfrac{b}{a}$. These are **self-conjugate diameters,** and will be shown to be **asymptotes**—*i. e.*, tangents at an infinite distance from the origin.

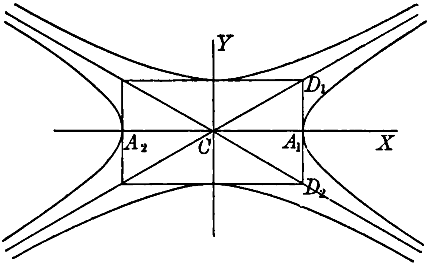
Also (Art. 284), $e=\sec A_1CD_1$ (fig. of Art. 358).

Proposition 24.

358. *Theorem.*—*The equation*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

represents the asymptotes of an hyperbola and its conjugate hyperbola.



For, the equation of the tangent line is $\frac{xx_1}{a^2}+\frac{yy_1}{-b^2}=1$. If $y=0$, $x_0=\frac{a^2}{x_1}$, which is the intercept on x. If $x=0$, $y_0=-\frac{b^2}{y_1}$, which is the intercept on y. If the co-ordinates of the point of tangency are $x_1=\infty$ and $y_1=\infty$, then $x_0=0$ and $y_0=0$. $\therefore$ the tangent at a point infinitely distant passes through the origin.

Also (Art. 342), $a_1^2=x_1^2+y_1^2$, $\therefore$ $a_1=\infty$, and then (Art. 357) $m=\pm\frac{b}{a}$, $\therefore$ $y=mx$, or $y=\pm\frac{b}{a}$ is at once a diameter and tangent at infinity. The same may be proved for the conjugate hyperbola. The equation $y=\pm\frac{b}{a}x$ may also be written

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0, \quad \text{or} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

359. *Schol.*—The *asymptotes* are the *diagonals of the rectangle* formed by the *tangents* at the *vertices*—*i. e.*, the rectangle of the axes.

360. *Exercise.*—Prove that $y=\pm\frac{b_1}{a_1}x$ is the equation of the asymptotes referred to conjugate diameters as axes.

Proposition 25.

361. *Theorem.*—*The equations*

$$xy = \pm \frac{a^2 + b^2}{4}$$

represent an hyperbola and its conjugate referred to the asymptotes as axes.

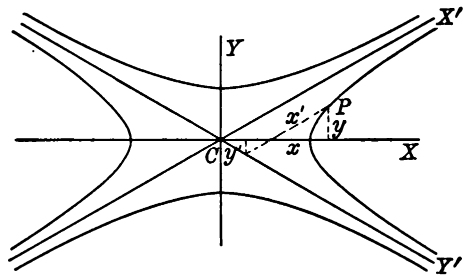
For, $\frac{x'}{x} = -\frac{y'}{x}$,

$\therefore \cos \frac{x'}{x} = \cos \frac{y'}{x}$

and

$\sin \frac{x'}{x} = -\sin \frac{y'}{x}$;

$\therefore$ (Art. 71)



$x = (x' + y') \cos \theta$

and

$y = (y' - x') \sin \theta.$

But (Art. 357) $\tan \theta = \frac{b}{a}$, $\therefore$ by trig., $\cos \theta = \frac{a}{\sqrt{(a^2 + b^2)}}$ and

$\sin \theta = \frac{b}{\sqrt{(a^2 + b^2)}}$; hence $x = \frac{a(x' + y')}{\sqrt{(a^2 + b^2)}}$ and $y = \frac{b(y' - x')}{\sqrt{(a^2 + b^2)}}$.

Substituting these values in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we have $$\frac{(x' + y')^2}{a^2 + b^2} - \frac{(y' - x')^2}{a^2 + b^2} = 1,$$

whence $$x'y' = \frac{a^2 + b^2}{4}.$$

Similarly, substitute in $\frac{x^2}{-a^2} + \frac{y^2}{b^2} = 1$ for the conjugate hyperbola,

$$\therefore \ x'y' = -\frac{a^2 + b^2}{4}.$$

362. ***Schol. 1.***—The equation $\frac{x}{x_1}+\frac{y}{y_1}=2$ represents the tangent at (x_1, y_1) referred to the asymptotes. For, the line through (x_1, y_1) and (x_2, y_2) is (Art. 90), $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$.

But $$x_1 y_1 = x_2 y_2 = \frac{a^2+b^2}{4}, \quad \therefore \quad y_2 = \frac{x_1 y_1}{x_2}$$

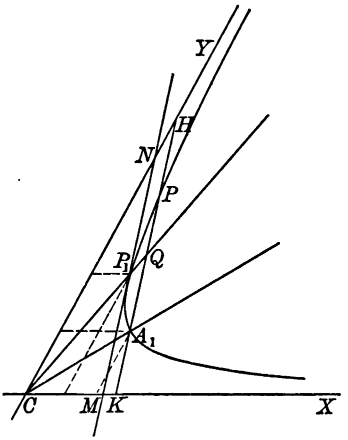
$$\therefore \quad \frac{y-y_1}{x-x_1} = \frac{\frac{x_1 y_1}{x_2} - y_1}{x_2 - x_1} = \frac{\frac{x_1 y_1 - x_2 y_1}{x_2}}{x_2 - x_1},$$

$$\therefore \quad \frac{y-y_1}{x-x_1} = \frac{y_1}{x_2}. \quad \text{Place } x_2 = x_1, \text{ then } \frac{x}{x_1}+\frac{y}{y_1}=2.$$

363. ***Schol. 2.***—In this, if $y=0$, $x_0=2x_1=$ intercept on x; and if $x=0$, $y_0=2y_1=$ intercept on y. $\therefore$ $x_0y_0=4x_1y_1=a^2+b^2$; $\therefore$ the product of the intercepts is *constant*.

364. ***Schol. 3.***—By trig., area CP_1 $=x_1y_1 \sin 2\theta = \frac{a^2+b^2}{4} \sin 2\theta$, since $x_1y_1=\frac{a^2+b^2}{4}$; $\therefore$ by trig., $x_1y_1 \sin 2\theta$ $=\frac{a^2+b^2}{2} \sin\theta \cos\theta = \frac{ab}{2}$ (Art. 361).

$\therefore$ area $CA_1=$ area $CP_1=\frac{ab}{2}$.



365. ***Schol. 4.***—Since $x_1=\frac{x_0}{2}$ and $y_1=\frac{y_0}{2}$, the point of contact (by sim. tri's) bisects MN, the portion of the tangent between the asymptotes.

Since CP_1 bisects MN, it bisects any parallel as HK at Q. $\therefore$ $HQ=QK$. But since CP_1 is a diameter, $PQ=QA_1$. $\therefore$ HP $=A_1K$. Hence to construct an hyperbola by points, draw a series of lines through A_1, and in each line make $HP=A_1K$, then the locus of P is the hyperbola.

366. ***Examples.***—Construct the loci represented by the following equations.

(1.) $\frac{x^2}{4}+\frac{y^2}{9}=1$, in which $\theta=60°$.

(2.) $4x^2-y^2=4$, in which $\theta=30°$.

(3.) $4y=13x^{-1}$.

(4.) $17x^{-1}y^{-1}=2$.

(5.) $y=4+3x^{-1}y-8x^{-1}$.

Find the equation of a tangent to each of the above loci at a point whose abscissa is 2.

Ans. (1.) $x=2$. (3.) $13x+16y=52$.

(2.) $y=\frac{4x}{\sqrt{3}}-\frac{2}{\sqrt{3}}$. (4.) $17x+8y=68$.

(5.) $x+y=11$.

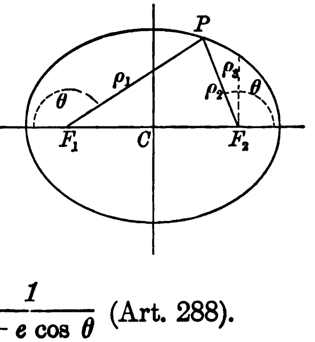
367. ***Exercise.***—Prove that the asymptotes are the diagonals of the parallelogram formed by drawing tangents through the vertices of any pair of conjugate diameters.

Proposition 26.

368. ***Theorem.***—*The equation*

$$\rho=\frac{l}{1+e\cos\theta}.$$

represents an ellipse or hyperbola; in which ρ *and* θ *are the polar co-ordinates,* θ *being measured from the nearest vertex, and* $l=\frac{\pm b^2}{a}$ *is the semi-latus rectum, the pole being at one of the foci.*



For, in the ellipse (Art. 307),
$\rho_1+\rho_2=2a$, and by trig.,
$\rho_1=\sqrt{\rho_2^2+4a^2e^2+4ae\rho_2\cos\theta}$.
Eliminate ρ_1, $\therefore$ ρ_2-2a
$=-\sqrt{\rho_2^2+4a^2e^2+4ae\rho_2\cos\theta}$.
Squaring and reducing,

$$\rho_2=\frac{a(1-e^2)}{1+e\cos\theta}=\frac{\pm b^2}{a}\cdot\frac{1}{1+e\cos\theta} \text{ (Art. 288).}$$

Let $\theta = \frac{\pi}{2}$, then, $\rho_3 = \frac{b^2}{a} = l$, $\therefore \quad \rho = \frac{l}{1 + e \cos \theta}$.

The same may be proved for the hyperbola.

369. ***Schol.***—The equation $\rho = \frac{a(1-e^2)}{1 \pm e \cos \theta}$ represents an ellipse, and the equation $\rho = \frac{\pm a(e^2-1)}{1 - e \cos \theta}$ represents an hyperbola, in which θ is measured as usual from $+x$, and the sign $+$ or $-$ is used, according as the right or left hand focus is the pole.

370. ***Examples.***—Construct the curves represented by the following equations.

(1.) $$\rho = \frac{4}{3 + \sqrt{5} \cos \theta}.$$

(2.) $$\rho = \frac{1}{2 + \tan \frac{\pi}{3} \cos \theta.}$$

(3.) $$\rho = \frac{4}{3 + \cos \theta + 4 \cos \theta \sin \frac{\pi}{10}}$$

Construct the polar of the point $(4, 6)$ with reference to the following loci.

(4.) $x^2 - y^2 = x + y + 4$. *Ans.* $y = \frac{2}{3}x - 1\frac{1}{3}$.

(5.) $x^2 + y^2 = 4$, in which $\omega = \frac{y}{x} = 60°$.

Ans. $2x + 3y = 2$.

(6.) Interpret the equation $x^2 - y^2 = x + y$.

371. ***Exercises.***—(1.) Prove as in the parabola that the focal polar equation of the tangent line is $\rho = \frac{l}{e \cos \theta + \cos (\theta - \theta_1)}$, in which θ is measured from the nearest vertex.

(2.) Prove by transformation that the central polar equation of the ellipse and hyperbola is $\rho = \frac{\pm b^2}{1 - e^2 \cos^2 \theta}$.

CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

Proposition 1.

372. ***Theorem.—The most general equation of the second degree, viz.:***

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.^* \quad \ldots \; (a.)$$

always represents one of the conic sections when A, B, C, F, G and H are any real constants.

The truth of this proposition will appear as the result of the succeeding propositions, in the following manner. We shall move the origin either to the centre or to the vertex, and then change the direction of the axes of x and y so that the axis of x shall coincide with the principal axis of figure of the curve; it will then appear that ($a.$) is reduced to a form identical with one of those before found to belong to the conic sections.

373. ***Schol. 1.***—The same result would follow whether ($a.$) be in *oblique* or *rectangular* co-ordinates. For, if it be in oblique co-ordinates, and be transformed to rectangulars, the co-efficients A, B, C, etc., will be changed, but the general form will remain the same (Art. 229). We shall, therefore, without the loss of generality, consider the case of rectangular axes.

374. ***Schol. 2.***—In this discussion A is supposed to be a positive quantity.

---

* We here adopt the coefficients in ordinary use at present. It may be useful to notice their symmetry as expressed in the square; *e.g.*, B is the coeff. of y^2, $2F$ of y, $2G$ of x, etc.

$$\begin{array}{c|ccc|} & x & y & 1 \\ \hline x & A & H & G \\ y & H & B & F \\ 1 & G & F & C \\ \hline \end{array}$$

Proposition 2.

375. ***Theorem.****—The equations*

$$x_0=\frac{BG-HF}{H^2-AB} \text{ and } y_0=\frac{AF-HG}{H^2-AB}$$

express the co-ordinates of the centre (x_0, y_0) *of the locus represented by equation* (***a.***).

For, let us move the origin to (x_0, y_0), whose co-ordinates are yet to be determined. From Art. 23,

$$x=x'+x_0, \quad \text{and} \quad y=y'+y_0,$$

$\therefore$ (***a.***) becomes

$$\begin{aligned}&Ax'^2+2Hx'y'+By'^2\\&+2(Ax_0+Hy_0+G)x'+2(By_0+Hx_0+F)y'\\&+Ax_0^2+2Hx_0y_0+By_0^2+2Gx_0+2Fy_0+C=0 \ldots (\textbf{\textit{b.}}).\end{aligned}$$

Now, if (x_0, y_0) is the *centre,* (***b.***) must be of such form as to remain unchanged, whether we substitute for x' and y', $+x'$ and $+y'$, the co-ordinates of one extremity of a diameter, or $-x'$, and $-y'$, the co-ordinates of the other extremity of the same diameter, since then, the origin evidently bisects the distance between (x', y') and $(-x,' -y')$. That such may be the case, there must be no terms of the first power in (***b.***)—*i. e.,*

$$Ax_0+Hy_0+G=0, \text{ and } By_0+Hx_0+F=0;$$

from which by elimination we find

$$x_0=\frac{BG-HF}{H^2-AB}\ldots(\textit{c.}), \text{ and } y_0=\frac{AF-HG}{H^2-AB}\ldots(\textbf{\textit{d.}}).$$

376. ***Schol. 1.***—If $H^2-AB=0$, the centre of the curve is at infinity—*i. e.*, it has no centre; but if $H^2-AB\lessgtr 0$, it has a centre.

377. ***Schol. 2.***—It may be noticed that the transformation resulting in (***b.***) does not change A, B or H, and that the new constant term

$$Ax_0+2Hx_0y_0+By_0+2Gx_0+2Fy_0+C=C' \ldots (\textbf{\textit{e.}})$$

is of the same form in x_0, y_0 as (***a.***) is in x, y.

378. ***Schol. 3.***—Hence the result of the transformation to the centre may be written

$$Ax'^2+2Hx'y'+Bx'^2+C'=0\ldots(f.)$$

379. ***Examples.***—Find the co-ordinates of the centres of the curves given in Art. 391, and the values of C'.

Proposition 3.

380. ***Theorem.***—*The equation*

$$\tan 2\theta=\frac{2H}{A-B}$$

expresses the value of the angle θ through which the co-ordinate axes x and y in equation (a.) must be turned to cause them to be parallel to the axes of the curve.

To turn the axis through any angle, θ, we place (Art. 80)

$$x=x''\cos\theta-y''\sin\theta,\ \text{ and }\ y=x''\sin\theta+y''\cos\theta.$$

Substitute in (*a.*) $\therefore$

$$(A\cos^2\theta+2H\sin\theta\cos\theta+B\sin^2\theta)\,x''^2$$
$$+2[(B-A)\sin\theta\cos\theta+H(\cos^2\theta-\sin^2\theta)]\,x''y''$$
$$+(A\sin^2\theta-2H\sin\theta\cos\theta+B\cos^2\theta)\,y''^2$$
$$+2(G\cos\theta+F\sin\theta)\,x''+2(F\cos\theta-G\sin\theta)\,y''+C=0\,..\,(g.).$$

This may be written

$$A'x''^2+2H'x''y''+B'y''^2+2G'x''+2F'y''+C=0\,..\,(h.).$$

Now, if $H'=0$, or $H(\cos^2\theta-\sin^2\theta)-(A-B)\sin\theta\cos\theta=0$,

then, $$\frac{H}{A-B}=\frac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}.$$

By trig., $\sin 2\theta = 2\sin\theta\cos\theta$, and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Substitute, $\therefore\ \dfrac{H}{A-B} = \dfrac{1}{2}\cdot\dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{1}{2}\tan 2\theta \ldots$ (*k.*).

Hence (*h.*) becomes

$$A'x''^2 + B'y''^2 + 2G'x'' + 2F'y'' + C = 0. \ \ldots \ (l.).$$

Now complete the square with respect to both x'' and y''.

$$\therefore\ A'\left(x'' + \frac{G'}{A'}\right)^2 + B'\left(y'' + \frac{F'}{B'}\right)^2 = \frac{G'^2}{A'} + \frac{F'^2}{B'} - C = -C',$$

or

$$\frac{\left(x'' + \dfrac{G'}{A'}\right)^2}{\dfrac{-C'}{A'}} + \frac{\left(y'' + \dfrac{F'}{B'}\right)^2}{\dfrac{-C'}{B'}} = 1. \ \ldots \ (m.),$$

which (Art. 296) represents an ellipse or hyperbola referred to axes parallel to the axes of the curve, when neither A' nor B' is equal to zero.

381. ***Schol. 1.***—By a similar transformation of (*f.*) its axes of x and y will become *coincident* with the axes of the curve; call the result (*g.*)$'$.
By trig., $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.
Substitute these relations and those above in (*g.*)$'$,

$$\begin{aligned}\therefore\ &\tfrac{1}{2}(A + B + 2H\sin 2\theta + (A-B)\cos 2\theta)x''^2 \\ &+ [(B-A)\sin 2\theta + 2H\cos 2\theta]x''y'' \\ &+ \tfrac{1}{2}(A + B - 2H\sin 2\theta - (A-B)\cos 2\theta)y''^2 + C' = 0.\end{aligned}$$

If $\tan 2\theta = \dfrac{2H}{A-B}$, then, as in Art. 380, $A'x''^2 + B'y''^2 + C' = 0$,

or

$$\frac{x''^2}{\dfrac{-C'}{A'}} + \frac{y''^2}{\dfrac{-C'}{B'}} = 1. \ \ldots \ (n.),$$

which is the central equation of some conic, if neither A' nor B' is equal to zero.

382. ***Examples.***—Find the values of θ for the curves given in Art. 391.

Proposition 4.

383. ***Theorem.***—*The equations*

$$H'^2 - A'B' = H^2 - AB \quad \text{and} \quad A' + B' = A + B$$

express relations which are invariable in any equation of the second degree, whatever may be the position of the axes.

For, evidently the transformation in Art. 375 does not change A, B or H, but that of Art. 380 gives us

$$A' = \tfrac{1}{2}[(A+B) + (A-B)\cos 2\theta + 2H\sin 2\theta],$$

$$2H' = (B-A)\sin 2\theta + 2H\cos 2\theta,$$

$$B' = \tfrac{1}{2}[(A+B) - (A-B)\cos 2\theta - 2H\sin 2\theta],$$

$$\therefore\ A' + B' = A + B. \ \ . \ . \ . \ (\textbf{p.}).$$

Also,
$$H'^2 - A'B' =$$

$$\tfrac{1}{4}[(A-B)\sin 2\theta - 2H\cos 2\theta]^2 - \tfrac{1}{4}(A+B)^2 + \tfrac{1}{4}[(A-B)\cos 2\theta + 2H\sin 2\theta]^2.$$

By trig.,
$$\sin^2 2\theta + \cos^2 2\theta = 1,$$

$$\therefore\ 4(H'^2 - A'B') = (A-B)^2 + 4H^2 - (A+B)^2,$$

$$\therefore\ H'^2 - A'B' = H^2 - AB. \ \ . \ . \ . \ (\textbf{q.}).$$

384. ***Schol. 1.***—Since, when (Art. 380) $\tan 2\theta = \dfrac{2H}{A-B}$, or $H' = 0$ the axes of x and y are parallel to those of the curve, (**q.**) then becomes

$$-A'B' = H^2 - AB. \ \ . \ . \ . \ (\textbf{r.}).$$

385. ***Schol. 2.***—If $\quad \tan 2\theta = \dfrac{2H}{A-B},$

by trig., $\quad \sin 2\theta = \dfrac{\tan 2\theta}{\sqrt{(1+\tan^2 2\theta)}}$, and $\cos 2\theta = \dfrac{1}{\sqrt{(1+\tan^2 2\theta)}}$.

Substitute, $\therefore\ \sin 2\theta = \dfrac{2H}{\sqrt{[(A-B)^2 + (2H)^2]}},$

and
$$\cos 2\theta = \frac{A-B}{\sqrt{[(A-B)^2 + (2H)^2]}}.$$

Substituting these values in the values of A' and B',

$$\therefore \quad A' = \tfrac{1}{2}[(A+B) + \sqrt{(A-B)^2 + (2H)^2}],$$

$$\text{and } B' = \tfrac{1}{2}[(A+B) - \sqrt{(A-B)^2 + (2H)^2}].$$

386. *Exercise.*—Prove that in Art. 380

$$C' = \frac{-ABC - 2FGH + AF^2 + BG^2 + CH^2}{H^2 - AB}.$$

Proposition 5.

387. *Theorem.*—*The invariant expression*

$$H^2 - AB$$

is also the criterion for determining which particular curve is represented by a given equation of the second degree.

The curve is an Ellipse if $H^2 - AB < 0$.

The curve is a Parabola if $H^2 - AB = 0$.

The curve is an Hyperbola if $H^2 - AB > 0$.

For, when $\tan 2\theta = \dfrac{2H}{A-B}$, then (Art. 384), $-A'B' = H^2 - AB$.

If A' and B' have *like* signs, then equation (*m.*) or (*n.*) evidently represents (Art. 288) an ellipse,
and then, $\quad -A'B' < 0, \therefore$ from (*r.*) $\qquad H^2 - AB < 0$.

If A' and B' have *unlike* signs, then eq. (*m.*) or (*n.*) represents (Art. 281) an hyperbola, and $-A'B' > 0$, $\therefore$ (*r.*) $H^2 - AB > 0$.

If, however, $-A'B' = H^2 - AB = 0$, then $A' = 0$, or $B' = 0$. Suppose $A' = 0$; by Art. 376, the centre is in this case at infinity, and equation (*l.*) becomes

$$B'y''^2 + 2F'y'' + 2G'x'' + C = 0,$$

or
$$\left(y'' + \frac{F'}{B'}\right)^2 = -\frac{2G'}{B'}\left(x'' + \frac{C}{2G'} - \frac{F'^2}{2B'G'}\right) \quad . \; . \; . \; (s.),$$

which (Art. 243) represents the parabola $y^2 = -\dfrac{2G'}{B'}x$, whose origin has been moved by using (Art. 23) equations,

$$x = x'' + x_0, \quad y = y'' + y_0,$$

when $$x_0 = \frac{B'C - F'^2}{2B'G'} \quad \text{and} \quad y_0 = \frac{F'}{B'}.$$

388. ***Schol. 1.***—If $$-A'B' = H^2 - AB < 0.$$
When $-\frac{C'}{A'} > 0$, and $-\frac{C'}{B'} > 0$, (*n.*) represents a *real* ellipse (Art. 288).
When $C' = 0$, the ellipse reduces to two imaginary right lines.

When $-\frac{C'}{A'} < 0$, and $-\frac{C'}{B'} < 0$, the ellipse is imaginary (Art. 301).

If $A' = B'$, the ellipse becomes a circle.

389. ***Schol. 2.***—If $$-A'B' = H^2 - AB = 0,$$
and $A' = 0$ only, (*s.*) represents a real parabola. If also $G' = 0$, then,
when $F'^2 - B'C > 0$, the parabola becomes two *real* parallel right lines;
when $F'^2 - B'C = 0$, the parabola becomes two *real* coincident right lines;
when $F'^2 - B'C < 0$, the parabola becomes two imaginary right lines, as may be shown from (*s.*).

390. ***Schol. 3.***—If $$-A'B' = H - AB > 0.$$
When $-\frac{C'}{A'} > 0$, and $-\frac{C'}{B'} < 0$, the hyperbola is primary (Art. 281).

When $C' = 0$, the hyperbola becomes two right lines.

When $-\frac{C'}{A'} <$, and $-\frac{C'}{B'} > 0$, the hyperbola is conjugate (Art. 299).

If $A' = -B'$, the hyperbola is rectangular.

391. ***Examples.***—Show what curves are represented by the following equations.

(1.) $4x^2 + 4xy + y^2 + 3x + 2y + 1 = 0.$

(2.) $2x^2 + 2xy + 2y^2 = x - y - 1.$

(3.) $y^2 + 8xy - 3x = 0.$

(4.) $3xy - x^2 - y^2 - x - y = 0.$

CHAPTER IX.

CURVES OF THE THIRD AND FOURTH DEGREES.

THIRD DEGREE.

Proposition 1.

392. ***Theorem.***—*The general equation of the third degree*

$$\begin{aligned}&A_1x^3 + \\ &A_2x^2 + B_1x^2y + \\ &A_3x + B_2xy + C_1xy^2 + \\ &A_4 \ + B_3y + C_2y^2 + D_1y^3 = 0 \ldots (a.)\end{aligned}$$

has been shown by Newton to be reducible in all cases to one of the following four forms:*

$$My + xy^2 = Ax^3 + Bx^2 + Cx + D \quad \ldots \ldots (b.)$$

$$xy = Ax^3 + Bx^2 + Cx + D \quad \ldots \ldots (c.)$$

$$y^2 = Ax^3 + Bx^2 + Cx + D \quad \ldots \ldots (d.)$$

$$y = Ax^3 + Bx^2 + Cx + D \quad \ldots \ldots (e.)$$

Included under (*b.*) are a large number of curves of various forms, all having at least two infinite branches. This may indeed be proved to be true of all curves of odd degree by Art. 215; for a curve of the n^{th} degree is cut in n points by one of the first degree—that is, by a straight line; and if cut in an odd number of points by a straight line, it cannot be composed entirely of finite loops.

---

*See Newton's "Enumeration of Lines of the Third Order."

Equation (*c.*) embraces but one form, called the *trident*, and equation (*e.*) represents one form only, called the *cubic parabola.*

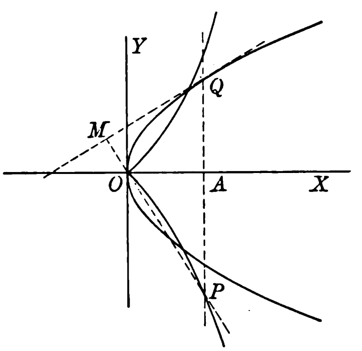
Those included under (*d.*) are of five species, and are called parabolas. It has been shown that the shadows of these five parabolas—that is, their conical projections—upon planes situated in various positions will give rise to all the other curves represented by (*a.*).

Proposition 2.

393. ***Theorem.—The semi-cubic Parabola***

$$y^2 = \frac{x^3}{p}$$

is the locus of P, the intersection of the ordinate QA of the parabola $y^2 = 4px$ with that perpendicular OM to the tangent QM, which passes through the vertex O.



For, if x_1 and y_1 are the coordinates of Q, and x' and y' of P, then (Art. 128)

$$y = -\frac{y_1}{2p}x \; . \; . \; . \; (f.)$$

is the perpendicular through O on the tangent (Art. 249)

$$yy_1 = 2p\,(x + x_1),$$

and $x = x_1 = x' \; . \; . \; . \; (g.)$

is the line QP. Combining (*f.*) and (*g.*), $\qquad y' = -\dfrac{y_1}{2p}x'$;

but $\qquad y_1 = \sqrt{4px_1} = \sqrt{4px'}; \quad \therefore \quad y' = -\dfrac{x'\sqrt{4px'}}{2p};$

$\therefore$ squaring and omitting the primes, $y^2 = \dfrac{x^3}{p}$.

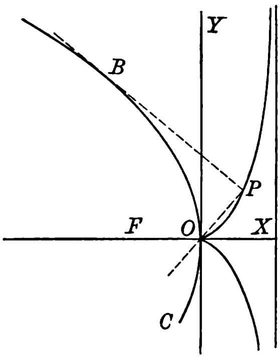
394. ***Exercise.***—The equations of the tangent and normal to this curve are, $y - y_1 = \dfrac{3x_1^2}{2py_1}(x - x_1)$, and $y - y_1 = -\dfrac{2py_1}{3x_1^2}(x - x_1)$.

Proposition 3.

395. ***Theorem.—The Cissoid***

$$y^2=\frac{x^3}{p-x}$$

is the locus of P, the foot of the perpendicular from the vertex O upon the tangents to the parabola $y^2=-4px$.



For, the equation of the tangent to $y^2=-4px$ is (Art. 256)

$$y=-mx+\frac{p}{m},$$

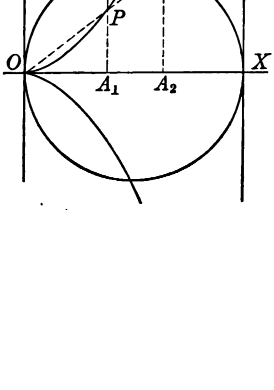
and the perpendicular to it through O is (Art. 128) $\quad y=\frac{x}{m},$

$$\therefore\quad m=\frac{x}{y}.$$

Eliminating, m, we have $\quad y^2=\frac{x^3}{p-x}.$

396. ***Schol.***—The polar equation of the cissoid is

$$\rho=\frac{p\sin^2\theta}{\cos\theta}=p\,(\sec\theta-\cos\theta).$$



397. ***Exercise.***—(1.) The cissoid $y^2=\frac{x^3}{2a-x}$ is the locus of P, the intersection of A_1B_1 and OB_2, when A_1B_1 and A_2B_2 are equal ordinates in the circle $y^2=2ax-x^2$.

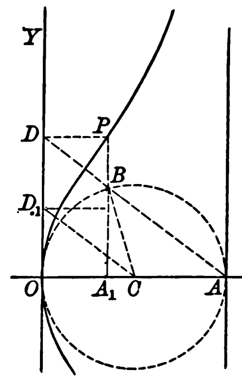
(2.) The cissoid $y^2=\frac{x^3}{2a-x}$ is the locus of P when $OB_2=PB$.

Proposition 4.

398. ***Theorem.—The Witch***

$$y^2 = \frac{4a^2x}{2a - x}$$

is the locus of P, a point on the linear sine at a distance from OX equal to twice the linear tangent of half the angle.



For, by hypothesis

$$y = 2a \tan \frac{\theta}{2} = 2a\sqrt{\frac{a(1 - \cos\theta)}{a(1 + \cos\theta)}},$$

$$\therefore \text{ by trig.,} \qquad y = 2a\sqrt{\frac{x}{2a - x}}.$$

399. ***Exercises.***—(1.) The witch $y^2 = \dfrac{4a^2x}{2a - x}$ is the locus of P when $A_1A : A_1B :: OA : A_1P = OD$.

(2.) The locus of P, whose distances ρ_1, ρ_2 and ρ_3 from three points P_1, P_2 and P_3 are such that either $2\rho_1 = \rho_2 + \rho_3$, or $\rho_1^2 = \rho_2\rho_3$, is a curve of the third degree.

Proposition 5.

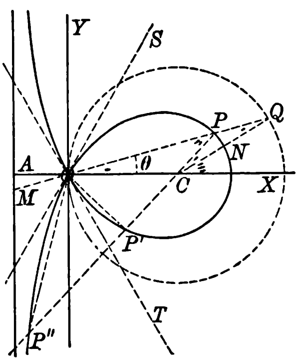
400. ***Theorem.—The cubic Trisectrix****

$$y^2 = \frac{x^2(3a - x)}{a + x}$$

is the locus of P upon the chord OQ of the circle whose radius is $2a$, such that $COQ = QCP$.

---

* My attention was first called to the properties of this curve as a trisectrix by the late Professor Wm. C. Cleveland.—AUTHOR.



For, by geometry the angles $COQ = CQO$ and $OPC = PQC + QCP = 2COQ$, and $XCP = OPC + COP = 3COQ$. $\therefore$ in the triangle COP

$$\rho : 2a :: \sin 3\theta : \sin 2\theta,$$

$$\therefore \quad \rho = 2a \frac{\sin 3\theta}{\sin 2\theta},$$

which is a polar equation of the curve.

If $AO = NQ = a$, then $PQ = MO$,

$\therefore \quad OP = OQ - PQ = OQ - MO;$

$\therefore \quad \rho = 4a \cos\theta - a \sec\theta$, which is another form of this polar equation. By transforming either of these polar equations by Art. 86 we obtain the above rectangular equation.

401. ***Schol.***—Let a line through C cut the curve in P, P' and P'', then $P'OP = P''OP' = 60°$. For, taking ρ in any position, as OP, if $XOP = \theta = \theta_1$, then $XCP = 3\theta$; and if a new position of ρ be taken, as OP', such that $\theta = \theta_1 \pm 60°$, then $XCP' = 3(\theta_1 \pm 60°) = 3\theta_1 \pm 180°$—that is, CP is in the same right line with CP'. Again, if another position of ρ be taken, such that $\theta = \theta_1 \pm 120°$, then $XCP'' = 3(\theta_1 \pm 120°) = 3\theta_1 \pm 360°$—*i. e.*, we have still the same line.

402. ***Exercise.***—Show that the polar equation of this curve with the pole at C is

$$\rho = a \sec \tfrac{1}{3}\theta,$$

and that the equation of the curve when the axes are the tangents OS and OT (in which $XOS = 60°$) is

$$x'^3 - 6ax'y' + y'^3 = 0.$$

Proposition 6.

403. ***Theorem.—The Folium***

$$y^2 = \frac{x^2(3a - x)}{3(a + x)}$$

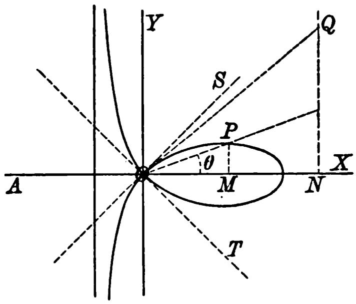
is the locus of P so situated with reference to two points A and O that (if $AO = 3a$) then $ON = 2OM = 2x$, and $OQ = AM = 3a + x$, and also $MOP = POQ$.

For, then $\dfrac{2x}{3a+x}=\cos 2\theta$;

$\therefore$ by trig.,

$$\frac{2x}{3a+x}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}},$$

$$\therefore\quad y^2=\frac{x^2(3a-x)}{3(a+x)}.$$



404. ***Schol.***—This is a projection of the trisectrix in which $y=y'/\sqrt{3}$.

405. ***Exercise.***—The equation referred to the tangents OS and OT as axes (in which $XOS=45°$) is of the form

$$x'^3-6bx'y'+y'^3=0.$$

Proposition 7.

406. ***Theorem.—The Logocyclic curve***

$$y^2=\frac{x(a-x)^2}{2a-x}$$

has the product of its two radii vectores constant.

For, transforming to polar co-ordinates by the equations

$$x=\rho\cos\theta,\quad\text{and}\quad y=\rho\sin\theta,$$

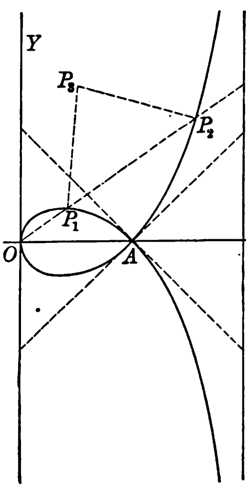
transposing, factoring and reducing by the relation $\sin^2\theta+\cos^2\theta=1$,

we have $\quad \rho^2-\dfrac{2a}{\cos\theta}\rho+a^2=0,$

whence $\quad \rho=a(\sec\theta\pm\tan\theta)$;

$\therefore\quad \rho_1\rho_2=a^2(\sec^2\theta-\tan^2\theta)=a^2$;

$\therefore$ also $OA=a$.



This curve has a parabola as the locus of the intersections of the normals drawn at P_1 and P_2.

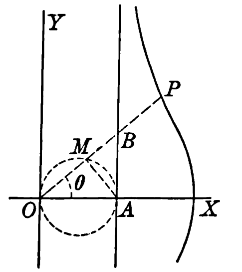
407. ***Exercise.***—The locus of P when $OP = OB + OM$ is (if $OA = a$)

$$\rho = a(\sec\theta + \cos\theta),$$

which is a curve of the third degree. This is one of the family

$$\rho = a(\sec\theta + n\cos\theta),$$

to which belong the cissoid and trisectrix.



FOURTH DEGREE

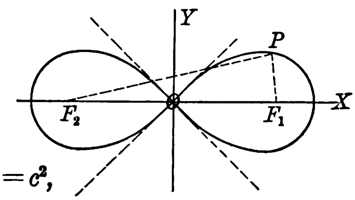
408. There are some thousands of curves of the fourth degree, a few of the more noted of which are discussed in the following propositions.

Proposition 8.

409. ***Theorem.—The Lemniscata***

$$(x^2+y^2)^2 = 2c^2(x^2-y^2) \quad \text{or} \quad \rho'\rho'' = c^2$$

is the locus of P moving so that the product of its focal radii, ρ' and ρ'', is equal to the square of half the distance between the foci.



For, let $OF_1 = F_2O = c$,

then $F_2P = \rho'' = [(x+c)^2 + y^2]^{\frac{1}{2}}$,

and $F_1P = \rho' = [(x-c)^2 + y^2]^{\frac{1}{2}}$,

$\therefore \quad [(x+c)^2+y^2]^{\frac{1}{2}}[(x-c)^2+y^2]^{\frac{1}{2}} = c^2,$

$\therefore \quad (x^2+y^2)^2 = 2c^2(x^2-y^2).$

410. ***Schol.***—If $OA = a$, the equation becomes

$$(x^2+y^2)^2 = a^2(x^2-y^2),$$

and the polar equation is

$$\rho^2 = a^2\cos 2\theta.$$

411. ***Exercise.***—Show that the polar equation of the ***Ovals of Cassini,*** which are defined by the equation

$$\rho'\rho'' = b^2,$$

in which ρ' and ρ'' are focal radii and b is any constant, is

$$\rho^4 - 2c^2\rho^2 \cos 2\theta = b^4$$

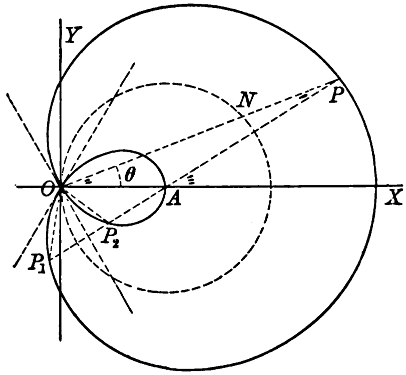
when the pole is halfway between the foci, whose distance apart is $2c$.

Proposition 9.

412. ***Theorem.—The Limaçon***

$$(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2)$$

is the locus of P at the intersection of two lines OP and AP moving so that $XAP = \frac{3}{2}XOP$.



For, in the triangle AOP, if $OA = a$, then

$$\rho : a :: \sin \tfrac{3}{2}\theta : \sin \tfrac{1}{2}\theta;$$

$$\therefore \quad \rho = a \frac{\sin \frac{3}{2}\theta}{\sin \frac{1}{2}\theta}$$

$$\therefore \quad \rho = a \frac{3 \sin \frac{1}{2}\theta - 4 \sin^3 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}$$

$$\therefore \quad \rho = a(3 - 4 \sin^2 \tfrac{1}{2}\theta)$$

$\therefore \quad \rho = a(1 + 2 \cos \theta)$, which equation, transformed by Art. 86, gives the above equation.

413. ***Schol. 1.***—The equation $\rho = a + 2a\cos \theta$ shows, since $ON = 2a \cos \theta$ (Art. 197), that $NP = a$.

414. ***Schol. 2.***—It may be shown in a manner similar to that used in Art. 401 that $POP_2 = P_2OP_1 = 60°$.

415. ***Exercise.***—(1.) Show that the polar equation of the limaçon with the pole at A is $\qquad \rho = 2a \cos \frac{1}{3}\theta.$

(2.) Construct the figure for the more general equation of the limaçon, of which the foregoing is a particular case, viz.:

$$(x^2 + y^2 - px)^2 = a^2(x^2 + y^2),$$

or
$$\rho = p \cos\theta \pm a;$$

when $p > a$ and when $p < a$.

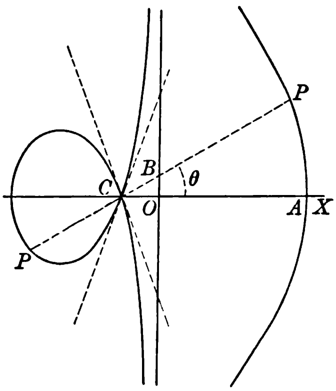
Proposition 10.

416. ***Theorem.—The Conchoid***

$$x^2y^2 = (a^2 - x^2)(b + x)^2$$

is the locus of P when $BP = \pm OA = a$, and $CO = b$.

For, then $\rho = b \sec\theta \pm a$, with the pole at C.



$$\therefore \quad \rho^2\left(1 - \frac{b \sec\theta}{\rho}\right)^2 = a^2,$$

$$\therefore \quad (x^2 + y^2)\left(1 - \frac{b}{x}\right)^2 = a^2,$$

$$\therefore \quad (x^2 + y^2)(x - b)^2 = a^2x^2.$$

Move the origin to O by making

$$x = x' + b,$$

then $x^2y^2 = (a^2 - x^2)(b + x)^2$.

417. ***Exercise.***—Draw the trifoliate curve $\rho = a \cos 3\theta$.

418. ***Definition.***—If from a pole O a perpendicular be let fall upon the tangent BP of any curve BC, the locus of the foot P of the perpendicular is the *pedal curve* with respect to the point O and the curve BC. *E. G.* The cissoid of Art. 395 is the pedal curve with respect to a parabola and its vertex.

Proposition 11.

419. ***Theorem.—The equation***

$$(x^2 + y^2 + x_1x)^2 = a^2x^2 \pm b^2y^2$$

represents the pedal curve of any conic and a pole on its

axis, referred to rectangular axes through the pole; in which x_1 is the distance of the pole from the centre of the conic, and a^2 and $\pm b^2$ are the squares of the semi-axes.

For, the equation of the tangent to any conic is (Art. 316) $y = m(x - x_1) + \sqrt{a^2m^2 \pm b^2}$, where x_1 is the distance of the origin from the centre of the conic, and $y = -\frac{x}{m}$ is a line through the origin perpendicular to this tangent, whence $m = -\frac{x}{y}$. Combine these equations, and we obtain the locus of their intersection,

$$y = -\frac{x^2 - xx_1}{y} + \sqrt{\frac{a^2x^2}{y^2} \pm b^2}$$

$$\therefore \quad (x^2 + y^2 - xx_1)^2 = a^2x^2 \pm b^2y^2.$$

420. ***Schol. 1.***—Let the pole be at the centre, and the conic an equiangular hyperbola—*i. e.*, $x_1 = 0$, and $a^2 = -b^2$;

$$\therefore \quad (x^2 + y^2)^2 = a^2(x^2 - y^2)$$

—that is, the curve is the lemniscata (Art. 409).

421. ***Schol. 2.***—Let the pole be at the vertex, and the conic a circle —*i. e.*, $x_1 = a$, and $a^2 = b^2$;

$$\therefore \quad (x^2 + y^2 - ax)^2 = a^2(x^2 + y^2),$$

which is a curve called a cardioid, whose polar equation is

$$\rho = a(1 + \cos\theta) = 2a\sin^2\tfrac{1}{2}\theta.$$

422. ***Schol. 3.***—Let the pole be at a distance $2a$ from the centre, and the conic a circle;

$$\therefore \quad (x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2),$$

which is the equation of the limaçon (Art. 412).

NOTE.—The student whose time is limited may complete the course from this point by taking four propositions in Chapter XI., or he may with greater advantage take three propositions in Chapter X., four or seven in Chapter XI., and three in Chapter XII.

CHAPTER X.

HIGHER ALGEBRAIC CURVES.

APPROXIMATE CURVES.

Proposition 1.

423. ***Theorem.—Parabolic Curves*** *represented by equations of the form $y^t = ax^r$, in which r and t are different positive integers, have the axis of x tangent to them at the origin when $r > t$, but the axis of y tangent at the origin when $r < t$.*

For, at any points (x_2, y_2) and (x_3, y_3) upon the curve we have

$$y_2^t = ax_2^r \quad \text{and} \quad y_3^t = ax_3^r.$$

Subtract, $\quad \therefore \quad y_2^t - y_3^t = a(x_2^r - x_3^r),$

$$\therefore \quad \frac{y_2 - y_3}{x_2 - x_3} = \frac{a(x_2^{r-1} + x_2^{r-2}x_3 + x_2^{r-3}x_3^2 + \ldots\ldots + x_3^{r-1})}{y_2^{t-1} + y_2^{t-2}y_3 + y_2^{t-3}y_3^2 + \ldots\ldots\ldots + y_3^{t-1}}.$$

But (Art. 90) $\quad \dfrac{y - y_3}{x - x_3} = \dfrac{y_2 - y_3}{x_2 - x_3}.$

Substitute, and then let $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$,

$$\therefore \quad \frac{y - y_1}{x - x_1} = \frac{arx_1^{r-1}}{ty_1^{t-1}}. \quad \ldots \quad (a.)$$

Then equation (*a.*) represents a tangent to $y^t = ax^r$ at (x_1, y_1).

In (*a.*) let $\quad x_1 = y_1 = 0.$

Whenever $\quad r > t \quad$ and hence $\quad r - 1 > t - 1,$

then, by algebra, $\dfrac{x_1^{r-1}}{y_1^{t-1}} = 0, \quad \therefore \quad$ equation (*a.*) becomes $y = 0,$

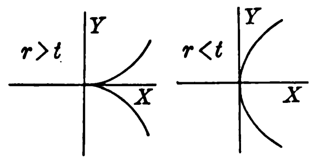
$\therefore$ (Art. 106) this tangent at the origin is the axis of x.

Whenever $r < t$ and hence $r - 1 < t - 1$,

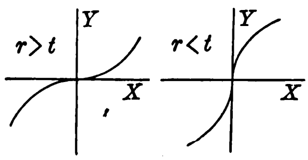
then by algebra $$\frac{x_1^{r-1}}{y_1^{t-1}} = \infty,$$

$\therefore$ (Art. 106) this tangent at the origin is the axis of y.

424. ***Schol. 1.***—When r is odd and t is even, the curve is symmetric about the axis of x (Art. 237, 1st), and has either of two forms near the origin according as $r > t$ or $r < t$, the former having a *cusp* at the origin.



425. ***Schol. 2.***—When r is odd and t is odd, the curve is symmetric in opposite quadrants (Art. 238, 2d), and there are two forms according as $r > t$ or $r < t$, both having a point of *inflexion* at the origin.



426. ***Schol. 3.***—When r is even and t is even, the equation may represent at least two separate curves. For, the equation, when both members are positive, takes the form $y^{2n} = b^2x^{2m}$.

Transpose and factor, $\therefore$ $(y^n + bx^m)(y^n - bx^m) = 0$.

$\therefore$ by algebra, $y^n = bx^m$ and $y^n = -bx^m$.

But when one member is positive and the other negative, the curve is imaginary.

When $r = t$, the equation represents one or more right lines.

Proposition 2.

427. ***Theorem.—Hyperbolic Curves*** ***represented by equations of the form*** $y^tx^r = a$, ***in which*** r ***and*** t ***are positive integers, all have both the axes of*** x ***and*** y ***tangent to them at infinity—i. e., both are asymptotes.***

For, at any points (x_2, y_2) and (x_3, y_3) upon the curve we have

$$y_2^t = ax_2^{-r} \quad \text{and} \quad y_3^t = ax_3^{-r}.$$

Subtract, $\therefore \quad y_2^t - y_3^t = a(x_2^{-r} - x_3^{-r}) = -\dfrac{a(x_2^r - x_3^r)}{x_2^r x_3^r}$,

$$\therefore \quad \frac{y_2 - y_3}{x_2 - x_3} = -\frac{a(x_2^{r-1} + x_2^{r-2}x_3 + x_2^{r-3}x_3^2 + \ldots\ldots + x_3^{r-1})}{x_2^r x_3^r (y_2^{t-1} + y_2^{t-2}y_3 + y_2^{t-3}y_3^2 + \ldots + y_3^{t-1})}.$$

Substitute in the equation of Art. 90, and then let

$$x_1 = x_2 = x_3 \quad \text{and} \quad y_1 = y_2 = y_3,$$

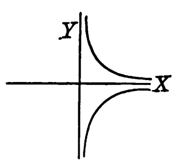
$$\therefore \quad \frac{y - y_1}{x - x_1} = -\frac{ar}{ty_1^{t-1}x_1^{r+1}}.$$

But $\qquad y_1^t = ax_1^{-r}, \quad \therefore \quad \dfrac{y - y_1}{x - x_1} = -\dfrac{ry_1}{tx_1}. \quad . \; . \; . \; (b.)$

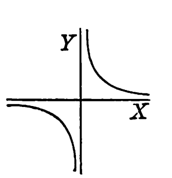
Equation (*b.*) represents a tangent to $y^t x^r = a$ at (x_1, y_1).
In (*b.*) let $x_1 = \infty$, then $y_1 = 0$ and $-\dfrac{ry_1}{tx_1} = 0$,
$\therefore$ (Art. 106) this tangent at $(\infty, 0)$ is the axis of x.

Again in (*b.*) let $y_1 = \infty$, then $x_1 = 0$ and $-\dfrac{ry_1}{tx_1} = \infty$,
$\therefore$ this tangent at $(0, \infty)$ is the axis of y.

428. ***Schol. 1.***—When r is odd and t is even, the curve is symmetric about the axis of x (Art. 237, 1st).



429. ***Schol. 2.***—When r is odd and t is odd, the curve is symmetric in opposite quadrants (Art. 238, 2d).



430. ***Schol. 3.***—When r is even and t is even, the equation may represent at least two separate curves (Arts. 236, 426).

431. ***Schol. 4.***—It appears from the nature of rectangular co-ordinates that when in the equation of any parabolic or hyperbolic curve discussed in this or the preceding proposition, y is written for x and x for y, the curve is thereby revolved *180°* about the line whose equation is $x = y$ —*i. e.*, about the bisector of the first angle.

But if $-y$ and $-x$ be written for x and y respectively, the curve is thereby revolved *180°* about the line $x = -y$ —*i. e.*, the bisector of the second angle. If $-y$ be written for y, the curve is thereby revolved *180°* about the axis of x; but if $-x$ be written for x, the curve is revolved *180°* about the axis of y.

Any combination of these replacements may be effected by performing them successively (cf. Arts. 237, 238).

Proposition 3.

432. ***Theorem.***—*1st. If a given curve has one or more branches through the origin (Art. 213), either a parabolic curve or right line may be found, one for each branch, which also passes through the origin, and which in shape and direction approximates to the branch near the origin.*

2d. If a given curve has infinite branches, either a parabolic curve, hyperbolic curve or right line may be found, one for each branch, which approximates to the position and direction of the branch at infinity.

This proposition is one of the general results of succeeding propositions in this chapter.

433. ***Schol.***—By moving the origin to different points of the given curve, the shape and direction of the curve may then be found by means of its less intricate approximate curves. Hence the approximate curves depend on the position of the origin.

EXPONENTIAL POLYGON.

434. The exponential polygon * is a device which enables us to find readily which of the terms in the equation of a curve of high degree may be neglected for the purpose of obtaining each of the simpler equations which represent curves approximating to the different branches of the given curve.

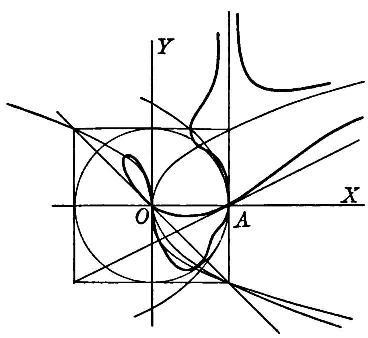
435. Exponential Axes. Draw any two lines OR and OT at right angles as exponential axes; and using the exponents r and t of any term ax^ry^t as the **exponential co-ordinates**, locate with respect to OR and OT the **representative point** (r, t); and in the same manner locate a representative point for each term of the given equation by using its exponents as co-ordinates.

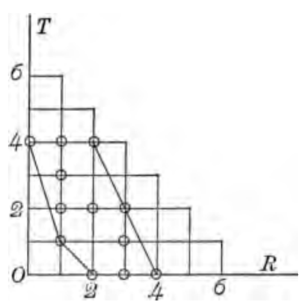
E. G. In the equation

$$(y^2 - ax)^2 (x - a)^2 - a^2xy (x^2 + y^2 - a^2) = 0,$$

or

$$x^2y^4 - 2ax^3y^2 + a^2x^4 - 2axy^4 + 4a^2x^2y^2 - 2a^3x^3$$
$$+ a^2y^4 - 2a^3xy^2 + a^4x^2 - a^2x^3y - a^2xy^3 + a^4xy = 0,$$





the term a^2y^4 has the representative point $(0, 4)$ marked by a small circle on OT. The term $-2ax^3y^2$ has the representative point $(3, 2)$,

* The exponential polygon here used is in principle the same as the "analytical triangle" and "parallelogram" employed by Newton and others.

and the other terms have their representative points as shown by the small circles marked upon the figure.

436. ***Remark.***—It will be seen that we have thus effected an arrangement of the terms of an equation in rank and file according to the powers of x and y in those terms, and that the representative points show only this, viz.: that terms containing certain powers of x and y occur in the equation; they in no sense represent points of the curve. In locating the representative points all coefficients are disregarded, since they in no way affect the position of those points.

437. ***Definition.***—Draw the smallest convex polygon, having representative points at its corners, which can contain within it or upon its sides all the representative points; this is the ***exponential polygon.***

438. Exponential Equations.—The sides of this polygon are straight lines, and may be conveniently represented by exponential equations such as $\frac{r}{a}+\frac{t}{b}=1$, which will facilitate our discussion of the polygon. The equation of the curve is assumed to contain no negative exponents, hence no representative points are on the negative side of either of the axes of r or t. There may be fractional exponents in the given equation, though in the cases we treat we shall assume the exponents to be integers.

It is also assumed that neither x nor y enters every term of the given equation; for if every term contain x, for instance, the equation is exactly divisible by x, and that factor may be removed. Since those terms which do not contain y have their representative points on OR, and those which do not contain x have theirs on OT, it is evident that the polygon has either a corner or a side in each of the axes of r and t.

439. Sides.—If each side of the exponential polygon be produced indefinitely, five kinds of sides may be distinguished, as follows:

1st. Sides whose intercepts on the axes of r and t are both positive, and which also lie between the origin and the rest of the polygon.

2d. Sides which have both intercepts positive, and which have the rest of the polygon between themselves and the origin.

3d. Sides whose intercepts are one positive and one negative; and also sides through the origin, all parallels to which have intercepts, one positive and one negative.

4th. Sides that coincide with either of the axes.

5th. Sides parallel to either of the axes, which have the rest of the polygon between themselves and the origin.

440. *N. B.*—If now from the given equation all terms be omitted except those whose representative points lie upon a single side of the exponential polygon, let us call the result "**the approximate equation for that side.**" And if from any equation all terms be omitted except those whose representative points lie on any assumed right line, let us call the result "**the approximate equation for that line.**"

441. The following proposition will be shown to be another of the general results of the succeeding propositions of this chapter.

Proposition '4.

Theorem.*—Each approximate equation for a side of the exponential polygon represents an approximate curve.*

When the approximate equation is for a side of the first kind, the approximation is near $(0, 0)$.

For a side of the second kind, the approximation is near (∞, ∞).

For a side of the third kind, the approximation is near $(0, \infty)$, or $(\infty, 0)$.

For a side of the fourth kind, the approximation is an intersection of the curve with the axis of x or y.

For a side of the fifth kind, the approximation is at infinity, and to a right line parallel to the axis of x or y.

Or, as it may be stated in other words, each approximate equation for a side of the exponential polygon represents an

approximate curve, when x is made infinite in all approximate equations for right-hand sides, and infinitesimal in those for all left-hand sides; while at the same time y is made infinite in all approximate equations for upper sides, and infinitesimal in those for all lower sides.

442. ***E. G.***—In the example, Art. 435, for sides of the first kind the equations are (see figure)

$$a^2y^4 + a^4xy = 0, \quad \text{or} \quad y^3 = -a^2x,$$

and
$$a^4x^2 + a^4xy = 0, \quad \text{or} \quad y = -x,$$

which represent, as will be shown, curves approximating to the branches of the given curve near the origin.

There is one side of the second kind from which

$$x^2y^4 - 2ax^3y^2 + a^2x^4 = 0,$$

whence $y^2 = ax$, which represents a curve which approximates to one of the infinite branches of the given curve.

There is one side of the fourth kind from which

$$a^4x^2 - 2a^3x^3 + a^2x^4 = 0,$$

whence $x = a$, which is the intersection of the curve with the axis of x. There is one side of the fifth kind from which

$$a^2y^4 - 2axy^4 + x^2y^4 = 0,$$

whence $x = a$, which is the equation of a straight line which approximates to one branch of the curve near (a, ∞).

Proposition 5.

443. *Theorem.—The approximate equation for any assumed right line is of the form*

$$y^t = ax^r$$

(r and t being positive integers), whenever the intercepts on that line upon the exponential axes are both positive; but is of the form

$$y^tx^r = a$$

whenever they are one positive and the other negative.

For, let $\frac{r}{r_0}+\frac{t}{t_0}=1$ be the equation of the assumed line referred to the axes of r and t, in which r_0 and t_0 are the intercepts. If r_1 and t_1 are the exponents of one term of the approximate equation for this line, r_2 and t_2 those for another, r_3 and t_3 those for another, etc., then we have the equations,

$$\frac{r_1}{r_0}+\frac{t_1}{t_0}=1, \quad \frac{r_2}{r_0}+\frac{t_2}{t_0}=1, \quad \frac{r_3}{r_0}+\frac{t_3}{t_0}=1.$$

Subtracting, transposing, etc.,

$$\frac{r_2-r_1}{t_2-t_1}=\frac{r_3-r_1}{t_3-t_1}=-\frac{r_0}{t_0} \quad \ldots \quad (a.),$$

which is the relation between the exponents of the terms whose representative points lie on this line—*i. e.*, the relation of the exponents in the general form of the approximate equation for this line.

This general form may be written

$$a_1x^{r_1}y^{t_1}+a_2x^{r_2}y^{t_2}+a_3x^{r_3}y^{t_3}+\ \ldots \ =0 \ \ldots \ (b.).$$

Divide this equation through by $x^{r_1}y^{t_1}$,

$$\therefore \quad a_1+a_2x^{r_2-r_1}y^{t_2-t_1}+a_3x^{r_3-r_1}y^{t_3-t_1}+\ \ldots \ =0,$$

or
$$a_1+a_2\left(x^{\frac{r_2-r_1}{t_2-t_1}}y\right)^{t_2-t_1}+a_3\left(x^{\frac{r_3-r_1}{t_3-t_1}}y\right)^{t_3-t_1}+\ \ldots \ =0,$$

which by equation (*a.*) becomes

$$a_1+a_2\left(x^{-\frac{r_0}{t_0}}y\right)^{t_2-t_1}+a_3\left(x^{-\frac{r_0}{t_0}}y\right)^{t_3-t_1}+\ \ldots \ =0,$$

from which, by algebra, one or more constant values of $x^{\frac{r_0}{t_0}}y$ can be obtained. In case r_0 and t_0 are both positive, $x^{\frac{r_0}{t_0}}y=c$ becomes $y^{t_0}=c^{t_0}x^{r_0}$, and is of the form $y^t=ax^r \ \ldots \ (c.)$.

But in case one intercept, as r_0, is negative,

let $\quad r_0=-r_0'$, then $x^{\frac{r_0'}{t_0}}y=c$, or $y^{t_0}x^{r_0'}=c^{t_0}$,

which is of the form $y^t x^r = a$ . . . (*d.*). We shall call the reduced equations (*c.*) and (*d.*) "*the* approximate equations," in distinction from (*b.*) "the general form of the approximate equation for any assumed line."

444. ***Schol.***—The approximate equations (*c.*) and (*d.*) for parallel lines can differ only in the value of the coefficient a.

For, in the equation $x^{-\frac{r_0}{t_0}} y = c$, the exponent, which is the negative ratio of the exponential intercepts, has, by similarity of triangles, the same value for all parallel lines. And conversely, all approximate equations that differ only in the value of the constant are for lines which are parallel. For, since the ratio of the exponential intercepts is the same (by similar triangles), the lines are parallel.

445. ***E. G.*** In the example, Art. 435, the general form of the approximate equation for the assumed line $\frac{r}{4} + \frac{t}{4} = 1$

is $$a^2x^4 - a^2x^3y + 4a^2x^2y^2 - a^2xy^3 + a^2y^4 = 0,$$

or $$x^4 - x^3y + 4x^2y^2 - xy^3 + y^4 = 0,$$

which is symmetrical in x and y, .

$\therefore$ by algebra $\frac{x}{y} = \frac{y}{x}$ and $\therefore$ $\frac{x}{y} = \pm 1$.

But $\frac{x}{y} = \pm 1$ does not satisfy the equation, therefore all values of $\frac{x}{y}$ are imaginary.

For the parallel to the assumed line, $\frac{r}{3} + \frac{t}{3} = 1$, the approximate equation is $\qquad -2a^3x^3 - 2a^3xy^2 = 0, \quad \therefore \quad x = y\sqrt{-1}.$

And for the other parallel, $\frac{r}{5} + \frac{t}{5} = 1$, we also obtain $x = y\sqrt{-1}$.

For the line $\frac{r}{2} + \frac{t}{-2} = 1$, the equation is

$$a^4x^2 - a^2x^3y = 0, \quad \therefore \quad xy = a^2;$$

and for the parallel line $\quad r = t \quad$ the equation is

$$4a^2x^2y^2 + a^4xy = 0, \quad \therefore \quad xy = 0 \quad \text{and} \quad xy = -\frac{a^2}{4}.$$

For the line $\frac{r}{2}+\frac{t}{8}=1,\quad a^4x^2-2axy^4=0;\quad \therefore\quad y^4=\frac{a^3x}{2};$

and for the parallel $\quad \frac{r}{3}+\frac{t}{12}=1,\quad x^2y^4-2a^3x^3=0;\quad \therefore\quad y^4=2a^3x.$

For the line $\quad \frac{r}{3}+\frac{t}{\frac{3}{2}}=1,\quad x^2=\frac{1}{2}ay;$

and for the parallel $\quad \frac{r}{5}+\frac{t}{\frac{5}{2}}=1,\quad x^2=-2ay;$

and for the parallel $\quad \frac{r}{7}+\frac{t}{\frac{7}{2}}=1,\quad x^2=-\frac{1}{2}ay.$

Proposition 6.

446. ***Theorem.***—*In order to compare the degrees of the several terms in the equation of any curve, replace x or y by its value obtained from an approximate equation, for any assumed line, of the form*

$$y^t=ax^r, \qquad \textit{then}$$

1st. *All terms whose representative points are on the assumed line become of the same degree.*

2d. *All the terms whose representative points are on any one line parallel to the assumed line become of the same degree.*

3d. *All terms whose representative points lie between the line and the origin become of less degree than those whose representative points are on the line, but all terms whose representative points are on the opposite side of the line become of greater degree.*

For, first, if in equation (*b.*) (Art. 443), we replace y by its value $cx^{\frac{r_0}{t_0}}$, then equation (*b.*) is reduced to the form

$$a_1'x^{r_1+\frac{r_0}{t_0}t_1}+a_2'x^{r_2+\frac{r_0}{t_0}t_2}+a_3'x^{r_3+\frac{r_0}{t_0}t_3}+\ldots=0\ \ldots\ (e.),$$

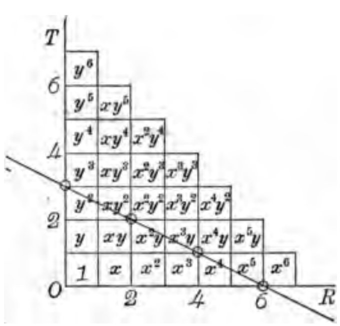
the degrees of whose successive terms, according to equation (*a.*), exceed that of the first term by

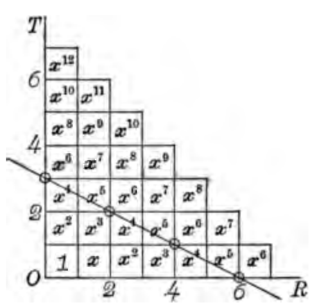
$$r_2-r_1+\frac{r_0}{t_0}(t_2-t_1)=0,\qquad r_3-r_1+\frac{r_0}{t_0}(t_3-t_1)=0,\text{ etc.}$$

And, second, since by Art. 444 all parallel lines have approximate equations which differ only in the value of the coefficient a, it is therefore evident that the expression (*e.*) above differs from that which we should get by replacing y with some value $c'x^{\frac{r_0}{t_0}}$ obtained from some parallel line—only in its coefficients.

But, third, since all the terms along any one of the parallel lines are of the same degree in x after the replacement, all the terms along each of the lines are of the same degrees respectively as the terms situated at the intersection of each of the lines with the axis of r. But the terms along r increase uniformly in degree from the origin; therefore each successive parallel counting from the origin has its terms of higher degree than the preceding.

447. ***Schol.***—This may appear more clearly if the facts be indicated upon a diagram. Let us write the literal part of every term in a





general equation of the sixth degree just at the right and above its representative point. Suppose the assumed line to be $\frac{r}{6}+\frac{t}{3}=1$;

$\therefore$ (equation (*c.*)), $y=ax^2$, for, $r_0=6$ and $t_0=3$,

and the terms thus become by the replacement of y as seen in the right hand figure.

And the same result would have appeared had we made use of any line the ratio of whose intercepts is the same.

448. *Cor.*—When one intercept is infinite (as t_0), we have

$$y = bx^{\frac{r_0}{\infty}} = b.$$

When $r_0 = t_0$, then $y = bx$.

Proposition 7.

449. *Theorem.*—*In order further to compare the degrees of the several terms in the equation of any curve, replace y by its value obtained from the approximate equation for any assumed line of the form $y^t x^r = a$ (the intercept on the axis of t being negative, and that on r positive); then,*

1st.* *All terms whose representative points are on the assumed line become of the same degree.

2d.* *All the terms whose representative points are on any one line parallel to the assumed line become of the same degree.

3d.* *All terms whose representative points lie* between *the line and the origin become of less degree than those whose representative points are* on *the line, but all terms whose representative points are on the* opposite *side of the line become of greater degree.

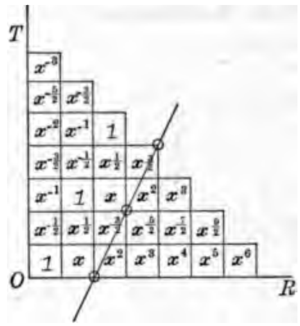
4th.* *If x be replaced instead of y, the terms whose representative points lie on the side toward the origin are of greater degree, and those on the opposite side of less degree.

5th.* *A corresponding statement is true when the intercept on the axis of r is negative, and that on t positive.

For, the first, second and third parts of the proposition are proved in the same manner as in Art. 446; and fourth, since $y^t x^r = a$, $\therefore$ $x = (ay^{-t})^{\frac{1}{r}}$, and $y = (ax^{-r})^{\frac{1}{t}}$—*i. e.*, the greater powers of x are the less powers of y, and vice versa.

450. ***Schol.***—This proposition may be illustrated as was the previous one. Assume the line

$$\frac{r}{2}+\frac{t}{-4}=1, \quad \therefore \quad y=\pm a x^{-\frac{1}{2}}.$$



451. ***Cor.***—When one intercept is infinite (as t_0), we have

$$x^{\frac{r}{\infty}} y=a, \quad \therefore \quad y=a.$$

452. ***Exercise.***—Write the literal parts of the terms in a similar manner when x is replaced.

Proposition 8.

453. ***Theorem.***—*An approximate equation of the form*

$$y^t=ax^r$$

for a side of the exponential polygon of the first kind (Art. 439) represents a curve which approximates to the original curve for infinitesimal values of x and y (i.e., near the origin); and an approximate equation for a side of the second kind is of the same form, but represents a curve that approximates to the original curve for infinite values of x and y.

For, if the general form of the approximate equation for a side of the first kind be obtained in the manner shown in Art. 443, equation (*b.*), and the value of y obtained from this equation be substituted in every term of the equation of the curve, it has been shown (Art. 446) that the terms whose representative points lie on this side are then of lower degree than the rest; for, this side lies between all others and the origin. When x is made infinitesimal after the replacement (*i. e.*, both x and y infinitesimal in the original equation), all other terms become vanishing quantities in comparison with those whose representative points lie on this side. In the same way it is evident that the terms whose representative points are on a side of the second kind are of higher degree than the rest, and that all other terms are vanishing quantities, when infinite values are assumed for x and y.

454. *Examples.*—Show that we have the following approximate curves.

(1.) In the Cissoid, $x^3 = py^2$, $x^2 + y^2 = 0$, $x = p$.

(2.) In the Witch, $y^2 = 2ax$, $x = 2a$.

(3.) In the Cubic Trisectrix, $y = \pm x\sqrt{3}$, $x = -a$.

(4.) In the Folium, $y = \pm x$, $x = -a$.

(5.) In the Lemniscata, and the equilateral hyperbola, $x = \pm y$.

(6.) In the Limaçon, $y = \pm x\sqrt{3}$.

(7.) In the Cardioid, $x^3 = -\frac{1}{2}ay^2$.

Proposition 9.

455. *Theorem.*—*An approximate equation of the form*

$$y^t x^r = a$$

for a side of the third kind represents a curve that approximates to the original curve for infinite values of x and infinitesimal values of y, when its intercept on the axis of r is positive, and that on t negative (that is, for that branch of the hyperbolic approximate curve to which the axis of x is an asymptote); but it represents a curve that approximates to the original curve for infinite values of y and infinitesimal values of x, when its intercept on the axis of t is positive, and that on r is negative.

For, the terms whose representative points lie on a side of the third kind are of higher degree in x and lower in y (or vice versa) than any other terms in the equation (Art. 449); therefore, if $x = \infty$ nearly, and $y = 0$ nearly, all other terms are vanishing quantities.

456. *Schol. 1.*—The general form of the approximate equation for a side of the fourth kind contains x only, or else y only. Suppose it is x only, then the other terms of the original equation vanish when $y = 0$. Hence we obtain the points of intersection of the curve with the axis of x.

457. *Schol. 2.*—The general form of the approximate equation for a side of the fifth kind contains the same powers of either x or y, according as the side is parallel either to the axis of r or t. Suppose each term has in it the same power of x, and on dividing through by that factor the equation will consist only of powers of y (*i. e.*, $y=a$ by Art. 451); then all the terms of the original equation whose representative points do not lie on this side vanish in comparison on making $y=a$ and $x=\infty$ nearly (*i. e.*, the curve approximates at infinity to the line $y=a$). A similar statement is true when $x=$ a constant.

Proposition 10.

458. *Theorem.—1st.* ***When an equation has no constant term—i. e., no representative point at (0, 0)—the curve has a single branch through the origin—i. e., it has a single point at (0, 0).***

2d. When an equation has no constant term nor terms of the first degree—i. e., no representative points at (0, 0), (0, 1) and (1, 0)—then the curve has two branches through the origin—i. e., it has a double point at (0, 0).

3d. When in addition it has no terms of the second degree, it has a triple point at the origin, etc., etc.

For, first, if there is no constant term in the equation, but there is a term of the first degree (containing y, say), then there is one, and but one, approximate equation of the form $y=ax^r$, as may be seen by finding the representative points of any such equation.

And, second, if there is a term of the second degree, but no constant term, and no terms of the first degree, it will appear from consideration of the exponential polygon either that there are two sides of the first kind, in which case the proposition must be true, or that in the approximate equation $y^t=ax^r$, $r=2$, or $t=2$. Suppose $t=2$, then $y=\pm\sqrt{ax^r}$; hence as x decreases to zero, there are two infinitesimal values of y, which vanish together. (It will be seen that a cusp is a double point as well as the intersection of two branches which cross at an angle.)

Also, third, from similar considerations the proposition may be extended to the case of a triple point, etc., etc.

459. ***Schol.***—If the origin be moved to any point of a curve, then an approximate equation for a side of the first kind may be found at that point which will show the direction of the curve at that point, and the kind of singularity, if any exists at that point. In general, the approximate equations are changed by transformation of co-ordinates. Review, Art. 432.

460. ***Examples.***—Discuss the following curves with reference to their multiple points at the origin, their approximate curves, and their intersections with the axes of x and y.

(1.) $y(y^2-1)=\left(x^2-\frac{9}{4}\right)\left(x^2-\frac{1}{4}\right).$

(2.) $y^4-96a^2y^2+100a^2x^2-x^4=0.$

(3.) $y^4+x^4=a^2(y^2-x^2).$

(4.) $x^2y^2-2axy-b^2(y^2+b^2)+2a^2b^2=0.$

(5.) $x^5-2a^3xy+y^5=0.$

(6.) $x^5-a^2x^2y-b^2xy^2+y^5=0.$

(7.) $\left(\frac{x}{a}\right)^{2n+1}+\left(\frac{y}{b}\right)^{2n+1}=1$, in which n is an integer.

(8.) $\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}=1$, when $n<\frac{1}{2}$, and when $n>1$, but not necessarily an integer. (If $n=\frac{1}{4}$ the curve is a parabola.)

(9.) Discuss the double points of the curves in Chapter IX.

461. ***Exercise.***—Move the origin in the example, Art. 435, to the point $(a, 0)$, and show that the transformed equation then has at $(0, 0)$ the approximate equations $x=2y$, and $y^2=-2ax$, and at $(0, \infty)$ the approximate equation $x^2y=a^3$.

462. ***General Scholium.***—The properties of the exponential polygon may also be used to discover the equation which will represent the general contour of a given curve whose equation is unknown.

CHAPTER XI.

TRANSCENDENTAL CURVES.

463. Transcendental Curves are those which require the use of transcendants, such as sin, tan, log, etc., to express the relation between the x and y of any point.

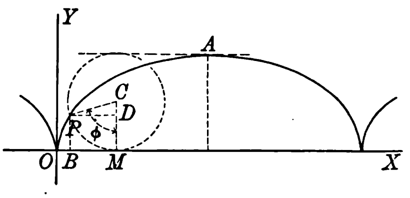
Several of these are discussed in the following propositions.

Proposition 1.

464. ***Theorem.***—*The equations*

$$x = r(\varphi - \sin\varphi) \quad \text{and} \quad y = r(1 - \cos\varphi)$$

represent the common cycloid, *which is the locus of a point on the circumference of a circle which rolls along a straight line; in which r is the radius of the circle, and φ is the angle of rotation.*



For, since $\varphi = PCM$

and $r = PC = MC$,

then $OM = \text{arc } MP = r\varphi$;

and if P is the point originally in contact with O,

then $x = OM - BM = r\varphi - r\sin\varphi = r(\varphi - \sin\varphi)$,

and $y = MC - DC = r - r\cos\varphi = r(1 - \cos\varphi)$.

465. ***Schol. 1.***—These equations are analogous to the equations of the ellipse and hyperbola in terms of the eccentric angle, and φ is called the *auxiliary angle* (Arts. 344, 349).

466. ***Schol. 2.***—As the circle continues rolling, an infinite number

of similar branches may be formed of which the equation of the n^{th} branch is

$$x = r[2n\pi + \varphi - \sin(2n\pi + \varphi)] = r(2n\pi + \varphi - \sin\varphi)$$
$$y = r[1 - \cos(2n\pi + \varphi)] = r(1 - \cos\varphi).$$

467. ***Exercise.***—Show that the first of the equations of Art. 464 may be written

$$x = r\cos^{-1}\left(\frac{r-y}{r}\right) - \sqrt{2ry - y^2},$$

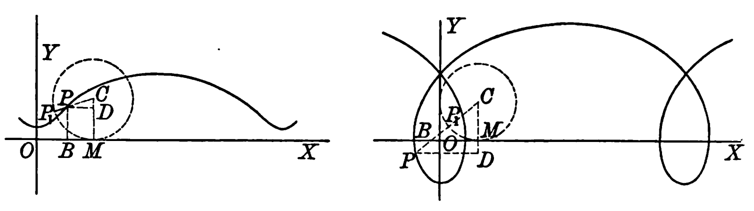
or
$$x = r\,\text{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}.$$

Proposition 2.

468. ***Theorem.***—*The equations*

$$x = r(\varphi - m\sin\varphi), \text{ and } y = r(1 - m\cos\varphi)$$

represent the locus of a point on the radius of a circle which rolls along a straight line; in which mr ***is the distance of the point from the centre of the circle, and the curve is a prolate or curtate cycloid according as*** $m < 1$ ***or*** $m > 1$. ***The curve is also called a trochoid.***



For, since $\varphi = PCM$, $mr = PC$ and $r = P_1C = MC$,

then $$OM = \text{arc } MP_1 = r\varphi,$$

then $$x = OM - BM = r\varphi - mr\sin\varphi = r(\varphi - m\sin\varphi),$$

and $$y = MC - DC = r - mr\cos\varphi = r(1 - m\cos\varphi).$$

The scholia of Proposition 1 apply to this proposition also.

469. *Exercise.*—Draw the **companion to the cycloid** which is determined by the equations $x = r\varphi$ and $y = r(1 - \cos\varphi)$.

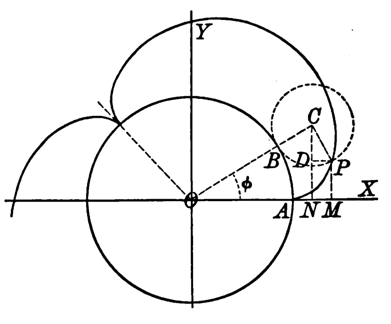
Proposition 3.

470. *Theorem.—The equations*

$$x = (r_1 + r_2)\cos\varphi - r_2\cos\left(\frac{r_1 + r_2}{r_2}\varphi\right)$$

$$y = (r_1 + r_2)\sin\varphi - r_2\sin\left(\frac{r_1 + r_2}{r_2}\varphi\right)$$

represent an epicycloid, which is the locus of a point on the circumference of a circle which rolls around on the outside of a fixed circle; in which r_1 is the radius of the fixed and r_2 that of the rolling circle, and φ its angle of revolution.



For, let $OB = r_1$, the radius of the fixed circle, and $CB = CP = r_2$, the radius of the rolling circle, when P is the point which was originally in contact with A. Let the angle of revolution $AOC = \varphi$, and the angle of rotation $OCP = \varphi'$,

then $\quad x = ON + NM = (r_1 + r_2)\cos\varphi + r_2\sin\left(\varphi + \varphi' - \frac{\pi}{2}\right),$

and $\quad y = NC - DC = (r_1 + r_2)\sin\varphi - r_2\cos\left(\varphi + \varphi' - \frac{\pi}{2}\right).$

$$\therefore\quad x = (r_1 + r_2)\cos\varphi - r_2\cos(\varphi + \varphi'),$$

$$y = (r_1 + r_2)\sin\varphi - r_2\sin(\varphi + \varphi').$$

But, since $\qquad r_1\varphi = r_2\varphi' = AB,$

eliminate φ', and we have the equations given above.

471. ***Schol.***—When $r_1 = r_2 = \frac{1}{2}a$, the equations become

$$\frac{x}{a} = \cos\varphi - \tfrac{1}{2}\cos 2\varphi = \cos\varphi(1 - \cos\varphi) + \tfrac{1}{2}$$

$$\frac{y}{a} = \sin\varphi - \tfrac{1}{2}\sin 2\varphi = \sin\varphi(1 - \cos\varphi).$$

Let $$x = \tfrac{1}{2}a - x', \quad \text{or} \quad \frac{x}{a} = \frac{1}{2} - \frac{x'}{a},$$

$$\therefore \quad -\frac{x'}{a} = \cos\varphi\,(1 - \cos\varphi),$$

$$\therefore \quad a^2(x'^2 + y^2) = a^4(1 - \cos\varphi)^2,$$

and $x'^2 + y^2 - ax' = a^2(1 - \cos\varphi)^2 + a^2\cos\varphi(1 - \cos\varphi) = a^2(1 - \cos\varphi)$,

$$\therefore \quad (x'^2 + y^2 - ax')^2 = a^2(x'^2 + y^2),$$

which is the equation of the cardioid (Art. 421).

472. ***Exercise.***—Show that the equations

$$x = (r_1 + r_2)\cos\varphi - mr_2\cos\left(\frac{r_1 + r_2}{r_2}\varphi\right)$$

$$y = (r_1 + r_2)\sin\varphi - mr_2\sin\left(\frac{r_1 + r_2}{r_2}\varphi\right)$$

represent the prolate and curtate epicycloid—*i. e.*, the epitrochoid.

Proposition 4.

473. ***Theorem.***—*The equations*

$$x = (r_1 - r_2)\cos\varphi + r_2\cos\left(\frac{r_1 - r_2}{r_2}\varphi\right)$$

$$y = (r_1 - r_2)\sin\varphi - r_2\sin\left(\frac{r_1 - r_2}{r_2}\varphi\right)$$

*represent an **hypocycloid,** which is the locus of a point on the circumference of a circle which rolls around the inside of a fixed circle.*

For, these equations may be obtained by a method similar to that used in the previous proposition, or by putting $-r_2$ for $+r_2$.

474. *Schol. 1.*—When $r_1 = 2r_2$, the equations become $x = 2r_2 \cos\varphi$, and $y = 0$, in which x may have any value between $+r_1$ and $-r_1$, and the curve is reduced to the diameter of the fixed circle.

475. *Schol. 2.*—When $r_1 = 4r_2$, we have

$$x = 3r_2 \cos\varphi + r_2 \cos 3\varphi = 4r_2 \cos^3\varphi$$

$$y = 3r_2 \sin\varphi - r_2 \sin 3\varphi = 4r_2 \sin^3\varphi,$$

$$\therefore \left(\frac{x}{r_1}\right)^{\frac{2}{3}} + \left(\frac{y}{r_1}\right)^{\frac{2}{3}} = 1,$$

one of the curves of the form given in Art. 460, Ex. (8).

It may be shown that the line $\frac{x}{a} + \frac{y}{b} = 1$ is always tangent to it when $a^2 + b^2 = r_1^2$—that is, on condition that this tangent line has a length $= r_1$ intercepted between the axes.

476. *Exercise.*—Show that the equations

$$x = (r_1 - r_2)\cos\varphi + mr_2 \cos\left(\frac{r_1 - r_2}{r_2}\varphi\right)$$

$$y = (r_1 - r_2)\sin\varphi - mr_2 \sin\left(\frac{r_1 - r_2}{r_2}\varphi\right)$$

represent the prolate and curtate hypocycloids—*i. e.*, the hypotrochoid —and also that when $r_1 = 2r_2$ the hypotrochoid becomes the ellipse

$$\frac{x^2}{(1+m)^2 r_2^2} + \frac{y^2}{(1-m)^2 r_2^2} = 1.$$

Proposition 5.

477. *Theorem.*—*The equations*

$$x = r(\cos\varphi + \varphi\sin\varphi) \quad \text{and} \quad y = r(\sin\varphi - \varphi\cos\varphi)$$

represent the involute* of a circle, which is the locus of P at the end of a thread unwound from the circumference of a circle whose radius is r.

* This involute is a spiral which has no real points within the generating circle.

For, if $XOB = DBP = \varphi$,

then $OBN = BPD = \frac{\pi}{2} - \varphi$,

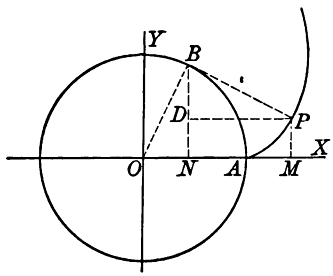
and $BP = AB = r\varphi$;

$\therefore\ x = OM = ON + DP$,

and $y = MP = NB - DB$;

$\therefore\ x = r \cos \varphi + r\varphi \sin \varphi$,

and $y = r \sin \varphi - r\varphi \cos \varphi$.



478. *Exercise.*—Show that the polar equation of this curve is

$$\sqrt{\frac{\rho^2}{r^2} - 1} = \theta + \cos^{-1}\frac{r}{\rho},$$

or

$$\sqrt{\frac{\rho^2}{r^2} - 1} - \tan^{-1}\sqrt{\frac{\rho^2}{r^2} - 1} = \theta.$$

TRIGONOMETRIC LOCI.

Proposition 6.

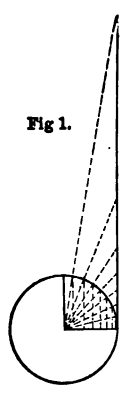
479. *Theorem.*—*The equations*

$$y = \sin x, \quad y = \tan x, \quad y = \sec x,$$

represent periodic or repeating curves.

For, let the curves be constructed by tracing them through points determined by tables of natural sines, tangents or secants, or by drawing the lines as in Fig. 1. We then have the curve of

Fig 1.



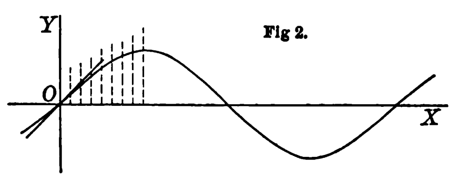


Fig 2.

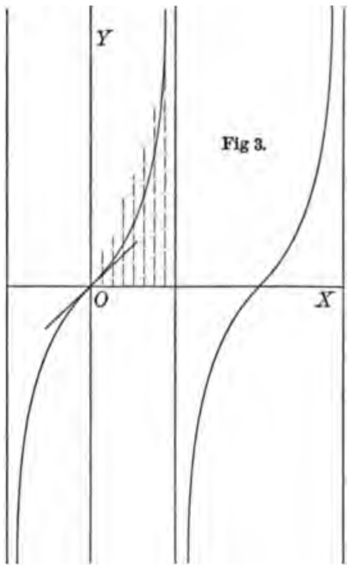
sines, Fig. 2; the curve of tangents, Fig. 3; and the curve of secants, Fig. 4.

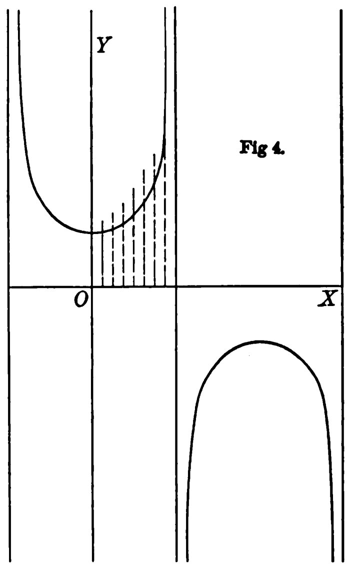
Since $\qquad \sin x = \sin\,(2n\pi + x),$

and $\qquad \tan x = \tan\,(2n\pi + x),$

and $\qquad \sec x = \sec\,(2n\pi + x)$

(when n is an integer), the curves repeat themselves along x.





480. ***Schol. 1.***—The equation

$$y = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \text{etc., by trig.,}$$

$\therefore$ the right line $y = x$ approximates to the curve at the origin (Art. 453)—*i. e.*, is tangent to it. Move the origin to $\left(\frac{1}{2}\pi,\ 1\right)$, $\therefore \ x = x' + \frac{1}{2}\pi$ and $y = y' + 1$, $\therefore \ y' + 1 = \sin\left(x' + \frac{1}{2}\pi\right)$ $= \cos x' = 1 - \dfrac{x'^2}{2} + \dfrac{x'^4}{24} - \text{etc., by trig.,}$ $\therefore$ the parabola $x'^2 = -2y'$ approximates to the curve at $\left(\frac{1}{2}\pi,\ 1\right)$. $\therefore$ also $y = \cos x$ is the same curve as $y = \sin x$ with a different origin, as is also $y = \text{versin}\ x$.

481. *Schol. 2.*—The equation

$$y = \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \text{etc., by trig.,}$$

$\therefore$ $y = x$ is tangent at the origin. This may be shown to be the same curve as $y = \cot x$ with a different origin.

482. *Schol. 3.*—The equation

$$y = \sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \text{etc.}}, \text{ by trig.,}$$

$$\therefore \quad y = \frac{1}{1 - \frac{1}{2}x^2 + \text{etc.}}, = 1 + \frac{x^2}{2} + \text{etc., by division,}$$

$\therefore$ at $(0, 1)$ the parabola $x^2 = 2y$ approximates to the curve. The curve $y = \text{cosec } x$ differs from $y = \sec x$ only in the position of the origin.

483. *Exercise.*—Construct the curve $x = \log y$ or $y = a^x$ when $a = 2$.

Proposition 7.

484. *Theorem.*—*If in the equation representing any locus (say, $\varphi(x, y) = 0$*), any trigonometric function of y be written in place of x, and also any trigonometric function of x in place of y, the equation thus obtained (say, $\varphi(\sin y, \tan x) = 0$) represents a **trigonometric locus** which may be readily traced from its relation to the original curve (viz., $\varphi(x, y) = 0$).*

The truth of this proposition will appear as the result of several succeeding propositions. The nature of the substitutions made may be seen from the following equations. From the equation of the right line $y = mx$ we thus obtain,

* This expression is read "ϕ function of x and y equal to zero."

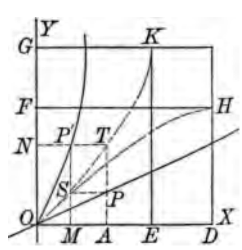
$$\sin x = m \sin y, \qquad \tan x = m \sin y,$$
$$\sin x = m \cos y, \qquad \tan x = m \tan y,$$
$$\sin x = m \tan y, \qquad \tan x = m \sec y,$$
$$\sin x = m \sec y, \qquad \tan x = m \cos y,$$
etc., etc. etc., etc.

From the equation of the parabola $y^2 = mx$ we thus obtain

$$\sin^2 x = m \sin y, \qquad \cos^2 x = m \tan y,$$
$$\tan^2 x = m \cos y, \qquad \sec^2 x = m \operatorname{versin} y,$$
etc. etc.

Proposition 8.

485. ***Theorem.***—*If the locus of the point P, which moves according to some law expressed by the equation $\varphi(x, y) = 0$, be traced, together with the auxiliary curves represented by the equations $x = \sin y$ and $y = \sin x$, then the trigonometric locus whose equation is $\varphi(\sin y, \sin x) = 0$ is the locus of the point P' situated at the corner of the rectangle $SPTP'$, whose sides are parallel to the axes of x and y, in which P is any point of $\varphi(x, y) = 0$, T is at the intersection of $x = \sin y$ with a line through P parallel to the axis of y, and S is at the intersection of $y = \sin x$, with a line through P parallel to the axis of x.*



For, let the x and y of the trigonometric locus be written x' and y'; then, if P is a point of any locus, as $y = mx$, and if S is the intersection of PS parallel to OX with OSH, the curve of sines whose equation is $y = \sin x'$, and if T is the intersection of PT parallel to OY with OTK, whose equation is $x = \sin y'$,

we have $\quad y = AP = MS, \quad \therefore \quad x' = OM = NP',$

and $\quad x = OA = NT, \quad \therefore \quad y' = AT = MP',$

$\therefore$ OP' is represented by $\sin x' = m \sin y'$. Hence, when x and y refer to P, x' and y' as above constructed refer to P', and the rectangle $SPTP'$ is always a construction of the relation between a locus and a trigonometric locus, derived from it by the previously mentioned substitution.

486. ***Schol. 1.***—Only that part of the locus $\varphi(x, y) = 0$ can be used in constructing $\varphi(\sin y, \sin x) = 0$, which lies within the square whose sides are $x = \pm 1$ and $y = \pm 1$, and every point P within this square has a corresponding point P' within the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$, while every point P which is on the side of the square $x = \pm 1$, $y = \pm 1$, has a corresponding point P' on the side of the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$.

487. ***Schol. 2.***—When PT is tangent to $\varphi(x, y) = 0$, then $P'T$ is also tangent to $\varphi(\sin y, \sin x) = 0$; for as P moves along its locus parallel to the axis of y, P' moves along its locus parallel to the axis of x. When PS is tangent, $P'S$ is likewise tangent. From this it follows that when the axis of x is tangent to $\varphi(x, y) = 0$ at the origin, the axis of y is tangent to $\varphi(\sin y, \sin x) = 0$ at the origin.

A single exception occurs; for when PT is tangent to $\varphi(x, y) = 0$, and PT is also the line KE or $x = \pm 1$, then $P'T$ is not tangent to $\varphi(\sin y, \sin x) = 0$, and similarly when PS is tangent, and is the line FH or $y = \pm 1$, then $P'S$ is not a tangent.

Proposition 9.

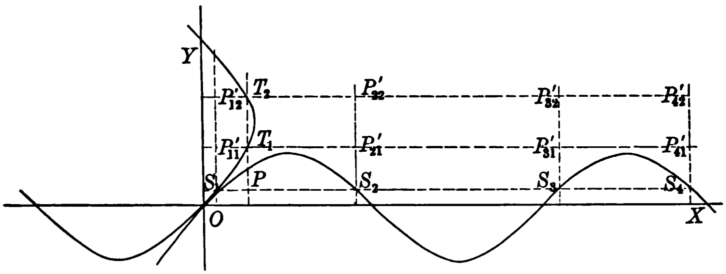
488. ***Theorem.***—*The equation*

$$\varphi(\sin y, \sin x) = 0$$

represents a curve (or calico pattern) such that one entire pattern which lies within the square whose sides are represented by the equations $x = \pm \frac{1}{2}\pi$ and $y = \pm \frac{1}{2}\pi$ is indefinitely repeated in each direction in such a manner as

would be caused by a kaleidoscope of four mirrors whose cross-section is the square mentioned.

For, it is noticeable that the line PS_1 intersects the curve $y = \sin x$ in an infinite number of points S_1, S_2, S_3, etc., on both sides of O.



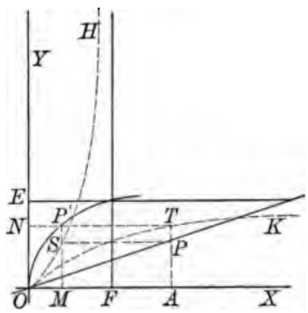
Also that PT_1 intersects the curve $x = \sin y$ in an infinite number of points T_1, T_2, T_3, etc., above and below O, and that from S_n and T_m we can obtain P'_{nm}. Again, as P is moved along, $\varphi(x, y) = 0$, P'_{11} moves away from the axes of y at the same rate that P'_{21} moves toward it, and vice versa; while P'_{12} moves away from the axis of x as P'_{11} moves toward it, and vice versa, causing the reflected repetition spoken of.

489. ***Schol.***—From Arts. 486 and 487, if $\varphi(x, y) = 0$ crosses $x = \pm 1$ or $y = \pm 1$ at any angle, then $\varphi(\sin y, \sin x) = 0$ crosses the sides of the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$ at right angles. That is, the patterns of the adjacent squares join in such a way as to have a common tangent parallel either to the axis of x or y. But if $\varphi(x, y) = 0$ is tangent to $x = \pm 1$ or $y = \pm 1$, then the corresponding part of $\varphi(\sin y, \sin x) = 0$ may meet $x = \pm \frac{1}{2}\pi$ or $y = \pm \frac{1}{2}\pi$ at any angle.

Proposition 10.

490. ***Theorem.***—*If* $\varphi(x, y) = 0$ *be traced and* P *be any point of it, and the auxiliary curves* $x = \tan y$ *and* $y = \tan x$ *be also traced, then* P' *the point of* $\varphi(\tan y, \tan x) = 0$ *corresponding to* P *may be constructed as in Proposition 8 by the use of the rectangle* $SPTP'$.

For, if P is any point of the locus $y=mx$, and if S is the intersection of PS parallel to OX with OSH whose equation is $y=\tan x'$, and if T is the intersection of PT parallel to OY with OTK whose equation is $x=\tan y'$,



then $\qquad y=AP=MS,$

$\therefore\ x'=OM=NP',$

and $\qquad x=OA=NT,$

$\therefore\ y'=AT=MP', \qquad \therefore\ OP'$ is the curve whose equation is $\varphi(\tan y, \tan x)=0$.

491. ***Schol. 1.***—All portions of $\varphi(x, y)=0$ are used in constructing $\varphi(\tan y, \tan x)=0$, and any finite point P has a corresponding point P' *within* the square $x=\pm\frac{1}{2}\pi$, $y=\pm\frac{1}{2}\pi$. P' is *on* the side of this square only when P is situated at infinity.

492. ***Schol. 2.***—When PT is tangent to $\varphi(x, y)=0$, then $P'T$ is tangent to $\varphi(\tan y, \tan x)=0$, and when PS is a tangent then $P'S$ is also a tangent (cf. Art. 487).

Proposition 11.

493. ***Theorem.***—*The equation*

$$\varphi(\tan y, \tan x)=0$$

represents a curve (or calico pattern) such that one entire pattern (which lies within the square $x=\pm\frac{1}{2}\pi$, $y=\pm\frac{1}{2}\pi$) is repeated indefinitely in each direction without change of form.

For, this may be made to appear by a demonstration precisely similar to that of Art. 488.

494. ***Schol.***—When $\varphi(x, y)=0$ has an infinite branch whose approximate curve at infinity is parabolic, then the corresponding part

of $\varphi(\tan y, \tan x) = 0$ crosses the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$ at a corner, and at an angle of $45°$ with the axes of x and y—*i. e.*, the patterns of the adjacent squares join in such a way as to have a common tangent $x = \pm y$. This appears from the fact that the auxiliary curves of tangents are asymptotic to the lines $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$.

Proposition 12.

495. ***Theorem.***—*By the aid of $\varphi(x, y) = 0$, and the auxiliary curves $x = \sin y$, and $y = \tan x$, the locus $\varphi(\sin y, \tan x = 0)$ may be constructed, one entire pattern of which lies within the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$, and is repeated indefinitely without change of form between the parallels $y = \pm \frac{1}{2}\pi$, the repetition being such as would be caused by a reflection in two plane mirrors whose cross-section is $y = \pm \frac{1}{2}\pi$.*

For, this may be made to appear by a mode of demonstration similar to that of the preceding propositions.

496. ***Schol.***—Only that part of $\varphi(x, y) = 0$ is used in constructing $\varphi(\sin y, \tan x) = 0$ which lies between the parallels $y = \pm 1$.

Proposition 13.

497. ***Theorem.***—*By the aid of $\varphi(x, y) = 0$, and the auxiliary curves $x = \sec y$, and $y = \sec x$, the locus $\varphi(\sec y, \sec x) = 0$ may be constructed, one entire pattern of which lies within the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$, and is repeated indefinitely in each direction in such a manner as would be shown by reflecting from four plane mirrors whose cross-section is $x = \pm \frac{1}{2}\pi$ and $y = \pm \frac{1}{2}\pi$.*

For, apply to this a demonstration similar to that of the previous propositions.

498. ***Schol.***—That part of $\varphi(x, y) = 0$ which lies within the square $x = \pm \frac{1}{2}\pi$, $y = \pm \frac{1}{2}\pi$ cannot be used in constructing $\varphi(\sec y, \sec x) = 0$.

499. ***Exercises.***—Construct the following repeating curves:

(1.) $$\sin^2 y = 4a \tan x.$$

(2.) $$\sin^2 x = \sin y - \sin^2 y.$$

(3.) $$\frac{\tan x}{\sin \alpha} + \frac{\sec y}{\sin \beta} = 1.$$

(4.) $$\frac{\sin^2 x}{\sin^2 \alpha} - \frac{\sin^2 y}{\sin^2 \beta} = 1$$

(5.) $$\cos^3 x + 3a \tan y \cos x + \tan^3 y = 0.$$

(6.) Construct and discuss the trig. loci obtained from combinations of trig. functions different from those already discussed.

E. G. $\varphi(\sec y, \tan x) = 0$, $\varphi(\sec y, \sin x) = 0$, etc., etc.

Trace the loci represented by the following equations, and also point out the manner in which the repetition occurs.*

(7.) $\sin x = 0$, $\sin x = 0.5$, $\sin x = 0.99$.

(8.) $\sin^2 x + \sin^2 y = a$, in which a is successively 0, 0.01, 0.99, 1, 2.

(9.) $\sin \rho = r$, in which r is successively 0, 0.01, 0.99.

(10.) $\sin \sqrt{x} = a$.

(11.) $\sin (\rho + x) = 0$.

(12.) $\sin (\rho + 1)^{-1} = 0$.

(13.) $y^2 = (1 - x^2) \cos nx$.

(14.) $\sin^2 x + \sin^2 y + \sin^2 \rho = 0$.

(15.) $\sin (y - \sin ax) = 0$.

(16.) $\sin (m \sin x \sin y) = 0$.

(17.) $\sin^{\frac{2}{3}} x + \sin^{\frac{2}{3}} y = \frac{1}{10} x$.

(18.) $\sin x \sin y = a \sin \frac{1}{10} x \sin \frac{1}{10} y$.

* Prof. H. A. Newton, of Yale College, has invented repeating curves in great number and variety. To him these equations are due.

CHAPTER XII.

SPIRALS AND POLAR CURVES.

500. A Spiral is the locus of a point moving along a line at the same time that the line itself revolves about a fixed point, provided that the two motions have such a relation that an infinite number of revolutions of the line must be made to complete the locus.

Frequently the polar equation of the spiral is in its simplest form when the fixed point is made the *pole*, the distance of the moving point the *radius vector* and the angle of revolution the *variable angle.*

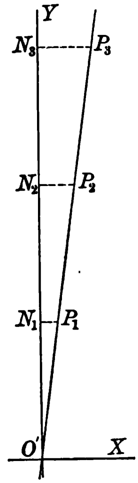
By a **polar curve** is usually signified one which has for its pole such a point as reduces its polar equation to some degree of simplicity. Such equations have been given for the trisectrix, limaçon, etc.

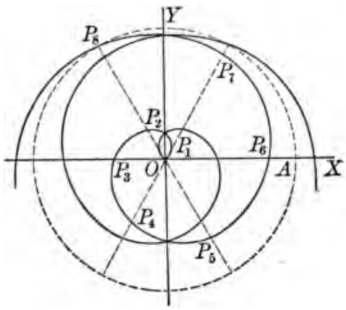
Proposition 1.

501. ***Theorem.***—*If in the equation of any algebraic curve (say, $\varphi(x, y) = 0$), ρ be written in place of x, and θ in place of y, the equation thus obtained (say, $\varphi(\rho, \theta) = 0$) represents a polar curve or spiral which may be readily traced from its relation to the original curve (viz. $\varphi(x, y) = 0$).*

For, the polar equation $\rho = a\theta$, by putting x for ρ and y for θ,

becomes $x = ay$. Let $a = \frac{1}{8}$ say. Draw the line $x = \frac{1}{8}y$. Also draw a circle with radius $OA = 1$, which is the *measuring circle*. The arc θ is measured off in linear units upon the circumference of this circle.





When $O'N_1 = y = \theta = \frac{1}{3}\pi$, then $N_1P_1 = x = \rho = \frac{1}{24}\pi$.

When $O'N_2 = y = \theta = \frac{2}{3}\pi$, then $N_2P_2 = x = \rho = \frac{1}{12}\pi$, etc., etc.

And a similar construction may evidently be made of any polar curve from its *analogous* rectangular curve.

502. ***Schol. 1.***—The polar curve $\rho = a\theta$ is the **spiral of Archimedes,** and it is analogous to the right line through the origin. In it the radius vector ρ is evidently proportional to the arc θ of the measuring circle. Since each value of θ negative as well as positive gives a single value of ρ, the spiral is symmetrical about the axis of y.

503. ***Schol. 2.***—The circle $\rho = a$ is analogous to the line $x = a$ parallel to the axis of y, and the fixed line $\theta = a$ is analogous to the line $y = a$ parallel to the axis of x.

504. ***Exercises.***—Construct the spirals,

(1.) $$\rho^3 - \frac{3}{2}\pi\rho\theta + \theta^3 = 0.$$

(2.) $$\left(\frac{\rho}{a}\right)^2 + \left(\frac{2\theta}{\pi}\right)^2 = 1.$$

(3.) $$\frac{\rho}{a} + \frac{2\theta}{\pi} = 1.$$

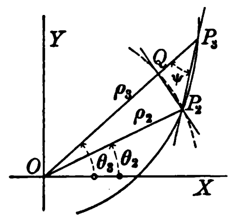
Proposition 2.

505. ***Theorem.—Parabolic Spirals*** ***represented by equations of the form***

$$\rho^r = a\theta^t,$$

in which r and t are different positive integers, all have the initial line tangent to them at the pole.

For, let P_2 and P_3 be upon the spiral, and let the angle $QP_3P_2 = \psi$. Then if P_2P_3 is infinitesimal, we may consider the perpendicular $QP_2 = \text{arc } \rho_2(\theta_3 - \theta_2)$ as it is when $\theta_2 = \theta_3$,



$$\therefore \quad \tan\psi = \frac{QP_2}{QP_3} = \frac{\rho_3(\theta_3 - \theta_2)}{\rho_3 - \rho_3}. \quad \ldots (a.)$$

when P_2P_3 is infinitesimal.

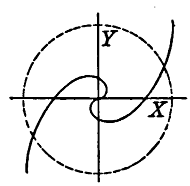
Find the value of $\dfrac{\theta_3 - \theta_2}{\rho_3 - \rho_2}$ as in Art. 423,

and then make $\quad \rho_1 = \rho_2 = \rho_3$ and $\theta_1 = \theta_2 = \theta_3$,

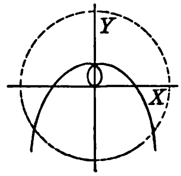
$$\therefore \quad \text{from } (a.) \quad \tan\psi = \frac{ar\rho_1^{\,r}}{t\theta_1^{\,t-1}} \quad \therefore \quad \tan\psi = \frac{r}{t}\theta_1,$$

in which ψ is the angle between the tangent and radius vector of (ρ_1, θ_1). Let $\theta_1 = 0$, then $\rho_1 = 0$, and $\tan\psi = 0$,
$\therefore$ the tangent to the spiral at the pole has the same direction as the initial radius vector and initial line.

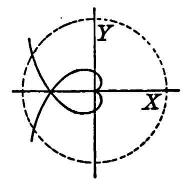
506. ***Schol. 1.***—When r is even and t is odd, the spiral has for every value of θ two numerically equal values of ρ with opposite signs, so that every polar chord is bisected at the pole.



507. ***Schol. 2.***—When r is odd and t is odd, the curve is symmetric about the axis of y, for $-\rho$ and $-\theta$ may be written for ρ and θ without altering the equation.



508. ***Schol. 3.***—When r is odd and t is even, the curve is symmetric about the axis of x, for equal values of θ of opposite sign give the same value of ρ.



509. ***Schol. 4.***—When r and t are both even, the equation may represent at least *two* of the previously mentioned spirals.

510. ***Schol. 5.***—The spiral whose equation is $\rho^2 = a\theta$ is frequently called *the* parabolic spiral.

511. ***Schol. 6.***—If $\theta_0 + \theta$ be put for θ (Art. 49) in the equation of any parabolic spiral, the curve is no longer tangent to the initial line, but is tangent instead to the line $\theta = -\theta_0$, for the curve is rotated through the angle $-\theta_0$.

Proposition 3.

512. ***Theorem.—Hyperbolic spirals represented by equations of the form***

$$\rho^r\theta^t = a,$$

in which r and t are positive integers, all approximate to the initial line at infinity, and approach the pole by an infinite number of revolutions of the generating line.

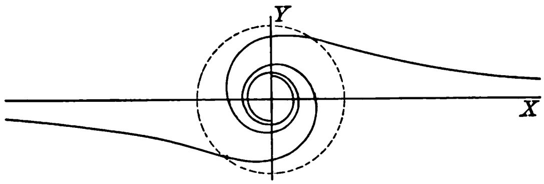
For, find the value of $\dfrac{\theta_3 - \theta_2}{\rho_3 - \rho_2}$, as in Art. 427, and substitute it

in (*a.*) of Art. 505, then make $\rho_1 = \rho_2 = \rho_3$ and $\theta_1 = \theta_2 = \theta_3$.

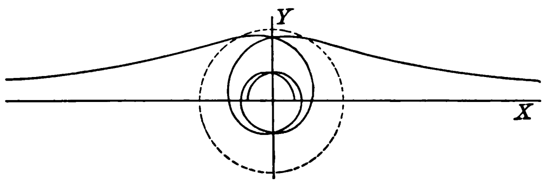
$$\therefore \quad \tan \psi = -\frac{r}{t}\theta_1 \ldots . (b.).$$

Let $\theta_1 = 0$, then $\rho_1 = \infty$, and $\tan \psi = 0$, $\therefore$ the spiral approximates to the initial line at infinity. Again, if $\rho = 0$, then $\theta = \infty$, $\therefore$ the spiral approaches the pole as θ becomes infinite.

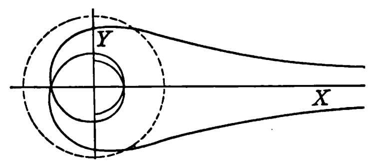
513. ***Schol. 1.***—When r is even and t is odd, each polar chord is bisected at the pole.



514. ***Schol. 2.***—When r is odd and t is odd, the spiral is symmetric about the axis of y.



515. ***Schol. 3.***—When r is odd and t is even, the spiral is symmetric about the axis of x.



516. ***Schol. 4.***—When r is even and t is even, the equation may represent two separate spirals.

517. ***Schol. 5.***—The spiral $\rho\theta = a$ is often called *the* hyperbolic spiral, and is asymptotic at infinity to a line parallel to the initial line and tangent to the circle $\rho = a$.

The Lituus $\rho^2\theta = a$ is asymptotic to the initial line at infinity, having the point of contraflexure $\left(\sqrt{2a}, \frac{1}{2}\right)$.

518. ***Schol. 6.***—If $\theta_0 + \theta$ be substituted for θ (Art. 49), in the equation of any hyperbolic spiral, the curve no longer approximates to the initial line at infinity, but approximates instead to the line $\theta = -\theta_0$.

519. The exponential polygon may be applied to finding spirals which approximate in shape and position to those parts of polar curves for which ρ and θ are infinite or infinitesimal.

The same principles are applicable to the determination of the approximate equations which can be derived from any given equation in ρ and θ, as were enunciated and proved in Chapter X. with respect to x and y, but the interpretation of the approximate equations so obtained needs consideration.

We shall state the interpretation of these approximate equations without formal proof, as the interpretation appears at once from the analogies already pointed out between polar and rectangular equations.

For convenience we shall consider that the spiral of Archimedes is one of the parabolic spirals.

520. Sides.—The exponential polygon, when applied to algebraic equations in ρ and θ, has sides of eight kinds, viz.:

1st. Sides whose intercepts on the axis of r and t are *both positive*, and which lie between the origin and the rest of the polygon.

The approximate equation for such a side represents a parabolic spiral which approximates to the curve of the original equation near the pole.

2d. Sides whose intercepts on the axis of r and t are *both positive*, and between which and the origin lies the rest of the polygon.

The approximate equation for such a side represents a parabolic spiral which approximates to the original curve for infinite values of ρ and θ.

3d. Sides whose intercepts are *positive* on the axis of r, and *negative* on the axis of t.

The approximate equation for such a side represents a hyperbolic spiral which approximates to the original curve for infinite values of ρ, and infinitesimal values of θ—*i. e.*, to the initial line at infinity.

4th. Sides whose intercepts are *negative* on the axis of r, and *positive* on the axis of t.

The approximate equation for such a side represents an hyperbolic spiral which approximates to the original curve for infinitesimal values of ρ and infinite values of θ—*i. e.*, to the *pole*.

A side whose intercepts are zero falls under the third or fourth case, according as all the rest of the polygon is above that side or to the right of it.

5th. Sides that coincide with the axis of r.

The approximate equation for such a side gives the value of ρ when $\theta = 0$—*i. e.*, the first intercept on the initial line.

6th. Sides that coincide with the axis of t.

The approximate equation for such a side gives the value of θ when $\rho = 0$—*i. e.*, the direction of the tangent at the pole.

7th. Sides parallel to the axis of r, and having the rest of the polygon between themselves and the origin.

The approximate equation for such a side gives a constant value of ρ when θ is infinite—*i. e.*, the circle to which the spiral is asymptotic.

8th. Sides parallel to the axis of t, and having the rest of the polygon between themselves and the origin.

The approximate equation for such a side gives a constant value of θ when ρ is infinite—*i. e.*, the direction of a radius vector which approximates to the position of an infinite branch of the spiral.

521. ***Examples.***—Find the spirals which approximate to
(1.) $\rho(\theta^2 \pm 1) = a\theta^2$. (2.) $\rho(\theta - \pi)^2 = a(\theta^2 - \frac{1}{4}\pi^2)$. (3.) $a^2\rho^2 = (\rho - a)^2\theta^2$.

522. ***Exercises.***—(1.) If a line OT be drawn from the pole O perpendicular to the radius vector $OP = \rho$, and intersecting at T the tangent TP, then $OT = \rho'$ is the *polar subtangent* of P, and (Arts. 505, 512) $\rho' = \rho \tan \psi$.

If $\rho = a\theta^t$, show that the locus of T is $\rho' = \frac{a}{t}(\theta \pm \frac{1}{2}\pi)^{t+1}$, in which t is positive, negative, integral or fractional.

(2.) Discuss the curves represented by the equation $\frac{\rho - a}{r} = \sin\frac{\theta}{n}$; they will illustrate the statements of Art. 234.

It will be found that when $r = a$ there are one or more cusps at the pole, and when $r < a$ the cusps are replaced by loops. *E. G.* When $n = 1$, if $r = a$ the curve is a cardioid, but if $r = 2a$ it is a limaçon. If $r > a$, the curve does not pass through the pole.

Again, it will be found that an entire branch of the curve is completed by the revolution of ρ through $2n\pi$. If n is *commensurable*, after the completion of a certain number of branches, no new branches are described by the further revolution of ρ. When n is a vulgar fraction in its lowest terms, the numerator states the number of revolutions of ρ before the figure is redescribed, and the denominator the number of similar branches in the complete curve. If n is *incommensurable*, both these numbers are *infinite*.

.1 Eddy, H.T. 623261

Analytic geometry.

NAME	DATE	NAME	DATE